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OUR CONTRIBUTORS

Edwin F. Beckenbach, Professor of Mathematics, University of California, Los Angeles, is a native of Texas and a graduate of Rice Institute (A.B.'28, A.M.'29, Ph.D.'31). A National Research Fellow from 1931-33, he returned to Rice as instructor in 1933, and later served on the faculties of Michigan and Texas Universities before coming to U.C.L.A. His wide mathematical interests include various domains of function theory and differential geometry, and he has published a number of articles in each of these fields. Dr. Beckenbach is an Associate Editor of the American Mathematical Monthly. His article "Metric Differential Geometry" appeared in the preceding issue of the Mathematics Magazine.

David V. Widder, Professor of Mathematics, Harvard University, was born in Harrisburg, Pennsylvania and is a Harvard graduate (A.B.'20, A.M.'23, Ph.D.'24). In the twenties, Dr. Widder was on the faculty of Bryn Mawr, serving as Head of the Mathematics Department from 1927-30, and was a National Research Fellow from 1926-27. A member of the Harvard faculty since 1931, he became Professor in 1937 and Department Chairman during 1942-48. He was stationed at Aberdeen Proving Ground, U.S.A., in the first World War and more recently has served as a civilian with the Office of Naval Research. An expert on integral transforms and series, Professor Widder is the author of "The Laplace Transform" Princeton 1941 and many papers on these subjects. He has also written a recent text on Advanced Calculus. He is the analysis editor for the Duke Mathematical Journal.

Joseph Seidlin, Dean, Graduate School, Alfred University is a native of Pavlograd, Russia, but received his college education in America at Columbia University (B.S.'15, M.A.'16, Ph.D.'31). After teaching at Rhodes and Lincoln Schools, Dr. Seidlin was appointed to the faculty of Alfred University in 1920 and became Professor in 1924, Department Head in 1933, Director of the Graduate Division in 1938, and Dean in 1947. His interests include geometry, teaching at the college and secondary levels and comparative education. The reader will recognize him as the editor of the Teaching Department of this Magazine.

Lewis Bayard Robinson was born in Baltimore, Maryland in 1883. He is by vocation an interior decorator associated with the Italian artist Angelo Pezzella in Baltimore, by avocation a mathematician working on functional equations and group theory.

Donald H. Hyers is a native of California and a graduate of U.C.L.A. (A.B.'33, M.A.'34) and the California Institute of Technology (Ph.D.'37). After teaching for five years at the University of Wisconsin he returned to Cal. Tech. on a war research project during 1942-44. He was appointed Associate Professor at the University of Southern California in 1944 and Department Head in 1945. His chief mathematical interest is in analysis, especially abstract spaces and differential equations.

AN INVERSION OF THE LAMBERT TRANSFORM

D. V. Widder

Introduction. A series introduced by J. L. Lambert [4; 507]* for use in his studies of the theory of numbers is the following,

$$(1) \qquad \sum_{n=1}^{\infty} \frac{a_n x^n}{1-x^n}.$$

It is natural to introduce an integral analogue of this series just as the Laplace integral may be presented as a continuous analogue of the power series. If in (1) we replace n by the continuous variable t, x by e^{-x} and Σ by $\int$ we obtain

$$(2) \qquad F(x) = \int_0^{\infty} \frac{a(t)}{e^{xt}-1}\, dt.$$

If we wish to consider (1) and (2) together we may use a Stieltjes integral (compare A. Wintner [12; 75])

$$(3) \qquad F(x) = \int_0^{\infty} \frac{d\alpha(t)}{e^{xt}-1}.$$

We shall refer to (2) or (3) as a Lambert transform. Since the denominator of the integrand vanishes at $t=0$, it is convenient to replace $a(t)$ by $t\,a(t)$ and $d\alpha(t)$ by $t\,d\alpha(t)$, thus introducing the continuous "kernal" $t/(e^{kt}-1)$. But for the heuristic description of the present section the forms (2) and (3) will be retained.

In §2 we discuss the convergence properties of the integral (3) and hence also of (1) and (2). They turn out to be just what one would expect from the known behavior of Lambert series (see, for example, [3]).

The remainder of the paper is devoted to the inversion of the transform (2). That of (3) could be derived therefrom in a familiar way [9]. Let us point out at once that a Lambert transform is a special case of a convolution transform

$$(4) \qquad f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt.$$

For, if we replace x by e^{-x} and t by e^{t}, equation (2) takes the form (4) with

*Numbers enclosed in brackets refer to the bibliography at the end of the present paper; numbers following a semicolon refer to the page of the article cited.

(5) $$G(x) = (e^{e^{-x}} - 1)^{-1}$$

$$f(x) = F(e^{-x}), \qquad \varphi(x) = e^{x} a(e^{x}).$$

The present author with I. I. Hirschman, Jr. [6,7,8,10,11] has discussed the inversion of (4) for a large class of kernels $G(x)$. However, the Lambert kernel (5) is not among those studied and must be considered *ab initio*. But the same operational methods used in [6] or [10] may be employed here to conjecture an inversion formula for (3).

The bilateral Laplace transform of the Lambert kernel (5) is well known [5; 45],

$$\int_{-\infty}^{\infty} e^{-st} G(t)\, dt = \int_{0}^{\infty} \frac{t^{s-1}}{e^{t}-1}\, dt = \zeta(s)\Gamma(s).$$

Let D stand for differentiation with respect to x, and make the usual interpretation of $e^{aD}\varphi(x)$ as $\varphi(x+a)$. If s is replaced by D in the above equation it becomes natural to interpret $\zeta(D)\Gamma(D)\varphi(x)$ as

$$\zeta(D)\Gamma(D)\varphi(x) = \int_{-\infty}^{\infty} \varphi(x-t)\, G(t)\, dt.$$

By making an obvious change of variable in the integral (4) this is seen to be $f(x)$, so that we should expect the desired inversion to have the symbolic form

(6) $$\varphi(x) = \frac{1}{\Gamma(D)} \frac{1}{\zeta(D)} f(x).$$

But it is known [5; 315] that

(7) $$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}} = \sum_{n=1}^{\infty} \mu(n) e^{-s\log n},$$

where $\mu(n)$ is the Möbius function. That is,

(8) $$\frac{1}{\zeta(D)} f(x) = \sum_{n=1}^{\infty} \mu(n)\, f(x - \log n).$$

On the other hand it has been pointed out repeatedly [11] that $1/\Gamma(D)$ is the symbolic operator for the inversion of the Laplace transform (after an exponential change of variable). Hence if (6) is to be the inversion desired, we should expect the right-hand side of (8) to be effectively a Laplace transform. We shall show, in fact, that

$$\sum_{n=1}^{\infty} \mu(n) F(nx) = \int_0^{\infty} e^{-xt} a(t)\, dt,$$

after which any of the known inversions of the Laplace transform will be effective in obtaining $a(t)$. The details of this operational development are given in §4.

The above considerations lead us to the following inversion of the transform (2),

$$a(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} \sum_{n=1}^{\infty} \mu(n) n^k F^{(k)}\left(\frac{kn}{t}\right). \tag{9}$$

We illustrate its use by the simple example $a(t) = t$. From (2) it is clear that $F(x)$ is $\zeta(2)/x^2$. But easy computations reduce the right-hand side of (9) to

$$\zeta(2) \lim_{k\to\infty} \frac{k+1}{k} \sum_{n=1}^{\infty} \frac{\mu(n)\, t}{n^2},$$

and by (7) this is seen to be t as predicted.

---

1. *Convergence.* We discuss first convergence of

$$F(x) = \int_0^{\infty} \frac{t\, d\alpha(t)}{e^{xt}-1} = \frac{1}{x} \int_0^{\infty} H(xt)\, d\alpha(t) \tag{1.1}$$

$$H(t) = \frac{t}{e^t - 1}, \quad 0 < t < \infty; \qquad H(0) = 1.$$

We assume that $\alpha(t)$ is a function of bounded variation in $0 \leqq t \leqq R$ for every positive R and that $\alpha(0) = 0$. In particular, if

$$\alpha(t) = \int_0^t a(u)\, du,$$

where $a(u)$ is a function which is integrable in the sense of Lebesque, then

$$F(x) = \frac{1}{x} \int_0^{\infty} H(xt)\, a(t)\, dt. \tag{1.2}$$

We shall refer to (1.2) as the *Lambert-Lebesque* transform to distinguish it from the *Lambert-Stieltjes* transform (1.1).

Since the kernel $H(xt)/x$, for positive x, differs very little from te^{-xt} when t is large it is natural to expect that the convergence behavior of (1.1) will be much the same as that of the Laplace integral

$$(1.3) \qquad \int_0^\infty e^{-xt}\, t\, d\alpha(t).$$

We shall show, in fact, that (1.1) and (1.3) converge for the same values of x if the abscissa of convergence σ_c for (1.3) is positive. For negative x, $H(xt)/x \sim -t$, and the similarity ends. Indeed when (1.3) converges at $x=0$, (1.1) converges for all $x \neq 0$. We prove the latter fact first.

Theorem 1.1. *If* (1.3) *converges at* $x=0$, *then* (1.1) *converges for all* $x \neq 0$.

Set

$$\beta(t) = \int_1^t u\, d\alpha(u), \qquad \beta(1) = 0,$$

so that $\beta(+\infty)$ exists by hypothesis. If $x \neq 0$, we have [9; 12]

$$\int_1^R \frac{t\, d\alpha(t)}{e^{xt}-1} = \int_1^R \frac{d\beta(t)}{e^{xt}-1} = \frac{\beta(R)}{e^{xR}-1} + x\int_1^R \frac{e^{xt}\beta(t)}{[e^{xt}-1]^2}\, dt \qquad R>1.$$

As $R \to +\infty$ the first term on the right tends to $-\beta(\infty)$ or to 0 according as $x<0$ or $x>0$. The integral on the right also tends to a limit since the integrand is $O(e^{-|x|t})$ as $t \to +\infty$. Hence we have established that (1.1) converges for all $x \neq 0$.

Theorem 1.2. *If* (1.3) *diverges at* $x=0$, *then the integrals* (1.1) *and* (1.3) *converge and diverge for the same values of* $x \neq 0$.

Let us first suppose that (1.3) converges at a point x_0 (which must be >0 by hypothesis). That is, if we set

$$\beta(t) = \int_1^t e^{-x_0 u}\, u\, d\alpha(u), \qquad \beta(1) = 0,$$

then $\beta(+\infty)$ exists, and

$$\int_1^R \frac{t\, d\alpha(t)}{e^{x_0 t}-1} = \int_1^R \frac{d\beta(t)}{1-e^{-x_0 t}} = \frac{\beta(R)}{1-e^{-x_0 R}} + x_0\int_1^R \frac{e^{-x_0 t}\beta(t)}{[1-e^{-x_0 t}]^2}\, dt \qquad R>1.$$

Clearly the right-hand side tends to a limit as $R \to +\infty$, so that (1.1) converges also at x_0.

Next suppose that (1.1) converges at a point x_0 (>0 or <0), and set

$$\gamma(t) = \int_1^t \frac{u\, d\alpha(u)}{e^{x_0 u}-1}, \qquad \gamma(1) = 0.$$

Then $\gamma(+\infty)$ exists, and

$$\int_1^R e^{-x_0 t} t\, d\alpha(t) = \int_1^R (1-e^{-x_0 t}) d\gamma(t) = \gamma(R)(1-e^{-x_0 R}) - x_0 \int_1^R e^{-x_0 t}\gamma(t)dt.$$

If $x_0 > 0$, the right-hand side approaches a limit as $R \to +\infty$. Hence (1.3) converges at x_0. If $x_0 < 0$, we have

$$\int_1^R t\, d\alpha(t) = \int_1^R (e^{x_0 t} - 1) d\gamma(t) = \gamma(R)(e^{x_0 R} - 1) - x_0 \int_1^R e^{x_0 t}\gamma(t)dt. \qquad R > 1$$

The right-hand side tends to a limit as $R \to +\infty$. But this conclusion is untenable in the presence of the hypothesis that (1.3) diverges at $x = 0$. Hence (1.1) must diverge for negative x, and (1.1) and (1.3) converge and diverge together for positive x.

2. *Relation to the Laplace transform.* In subsequent work we deal only with the Lambert-Lebesque transform

$$F(x) = \frac{1}{x}\int_0^\infty H(xt)\, a(t)\, dt \tag{2.1}$$

where $a(t)$ is Lebesque integrable on $(0,\infty)$,

$$\int_0^\infty |a(t)|\, dt < \infty. \tag{2.2}$$

Of course assumption (2.2) permits $a(t)$ to become infinite as $t \to 0+$ but certainly not so strongly as $1/t$. For simplification of subsequent computations we make the following explicit assumption about the behavior of $a(t)$ near $t = 0$.

$$\lim_{t\to 0+} a(t)\, t^{1-\delta} = 0 \tag{2.3}$$

for some positive number δ. This condition is equivalent to

$$\overline{\lim_{t\to 0+}} \frac{\log |a(t)|}{\log (1/t)} < 1. \tag{2.4}$$

We consider also the corresponding Laplace-Lebesque transform

$$f(x) = \int_0^\infty e^{-xt}\, t\, a(t)\, dt, \tag{2.5}$$

and study first how $F(x)$ may be expressed in terms of $f(x)$.

Theorem 2.1. *If* $F(x)$ *and* $f(x)$ *are defined by* (2.1), (2.2), (2.3), (2.5), *then*

$$F(x) = \sum_{k=1}^{\infty} f(kx), \tag{2.6}$$

the series converging absolutely for $x > 0$.

Observe first that

$$H(t) = \frac{t}{e^t - 1} = t \sum_{k=1}^{\infty} e^{-kt} \qquad 0 < t < \infty$$

and that

$$0 < H(t) \leqq 1 \qquad 0 \leqq t < \infty. \tag{2.7}$$

If the series

$$\frac{1}{x} H(xt)\, a(t) = \sum_{k=1}^{\infty} t\, a(t) e^{-kxt}$$

may be integrated term by term with respect to t from 0 to ∞ we shall have equation (2.6) at once. This will be permissible if

$$\int_0^{\infty} t\, |a(t)| \sum_{k=1}^{\infty} e^{-kxt}\, dt < \infty.$$

But for $x > 0$ this integral is

$$\int_0^{\infty} \frac{t\, |a(t)|}{e^{xt} - 1}\, dt = \frac{1}{x} \int_0^{\infty} H(xt)\, |a(t)|\, dt,$$

and by (2.7)

$$\int_0^{\infty} H(xt)\, |a(t)|\, dt < \int_0^{\infty} |a(t)|\, dt.$$

By our hypothesis (2.2) the result is established.

We prove that every Lambert transform (2.1), (2.2), (2.3) is also a Laplace transform.

Theorem 2.2. *If* $F(x)$ *is defined by* (2.1), (2.2), (2.3), *then*

$$F(x) = \int_0^{\infty} e^{-xt}\, t\, b(t)\, dt \qquad 0 < x < \infty,$$

where

$$b(t) = \sum_{k=1}^{\infty} \frac{1}{k^2}\, a\left(\frac{t}{k}\right),$$

the series converging absolutely for $0 < t < \infty$.

From the previous theorem we have

$$F(x) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-kxt}\, t\, a(t)\, dt = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-xt} \frac{t}{k^2}\, a\left(\frac{t}{k}\right) dt.$$

Our result will be established if we may interchange integral and summation signs, and this will be permissible if

$$\sum_{k=1}^{\infty} \int_0^{\infty} e^{-xt} \frac{t}{k^2} \left|a\left(\frac{t}{k}\right)\right| dt < \infty.$$

This series is equal to

$$\sum_{k=1}^{\infty} \int_0^{\infty} e^{-kxt}\, t\, |a(t)|\, dt = \int_0^{\infty} \frac{t\,|a(t)|}{e^{xt}-1}\, dt$$

provided either side of the latter equation is finite. But we established this fact for the right-hand side in the proof of the previous theorem.

Conversely, the Laplace integral (2.5), (2.2), (2.3) can also be expressed in terms of the Lambert integral (2.1). To show this we need to introduce the Möbius function $\mu(n)$ mentioned in the introduction. It is defined to be 1 when $n=1$, 0 when n is divisible by a square, and $(-1)^k$ when n is the product of k distinct primes. An alternative definition is

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

If we multiply these two series together we verify easily the familiar relation

(2.8) $$\sum_{d/n} \mu(d) = 0 \qquad n = 2, 3, \cdots,$$

where the summation runs over all the divisors d of n (including 1 and n).

We shall need the following preliminary result.

Lemma 2.3. *If the double series*

(2.9) $$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mu(n)\, f(knx)$$

converges absolutely, it has the falue $f(x)$.

For then it may be summed in any manner. If we group all terms involving $f(mx)$, the coefficient will be $\sum_{d/m} \mu(d)$, and the double series

$$\sum_{m=1}^{\infty} f(mx) \sum_{d/m} \mu(d).$$

By (2.8) this is equal to $f(x)$.

Theorem 2.4. *If* $f(x)$ *is defined by* (2.5), (2.2), (2.3), *then*

$$f(x) = \int_0^{\infty} \frac{t\,c(t)}{e^{xt}-1}\,dt \qquad 0 < x < \infty,$$

where

$$c(t) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\, a\left(\frac{t}{n}\right),$$

the series converging absolutely for $0 < t < \infty$.

By the definition of $f(x)$ we have for any positive integers n and k

$$f(knx) = \int_0^{\infty} e^{-kxt}\, \frac{t}{n^2}\, a\left(\frac{t}{n}\right) dt \qquad 0 < x < \infty.$$

Since $|\mu(n)| \leqq 1$, the absolute convergence of the double series (2.9) will be established for the present function (2.5) if

$$(2.10) \qquad \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-knxt}\, t\,|a(t)|\, dt < \infty.$$

By hypothesis (2.3) there exist constants M, g and δ such that

$$|t\,a(t)| \leqq M\,t^{\delta} \qquad 0 \leqq t \leqq g.$$

Hence

$$(2.11) \quad \int_0^{\infty} e^{-xt}\, t|a(t)|\, dt \leqq M\int_0^{g} e^{-xt}\, t^{\delta}\, dt + \int_g^{\infty} e^{-xt}\, t|a(t)|\, dt$$

$$\leqq M\int_0^{\infty} e^{-xt}\, t^{\delta} dt + g\, e^{-gx} \int_g^{\infty} |a(t)|\, dt$$

when $x > 1/g$. That is, the integral on the left of (2.11) is $O(x^{-\delta-1})$ as $x \to +\infty$. Consequently the series (2.10) is dominated by

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{N}{(knx)^{\delta+1}} = \frac{N\zeta^2(1+\delta)}{x^{\delta+1}},$$

where N is a suitable constant. By Lemma 2.3 the sum of the series (2.9) is $f(x)$, and by (2.10) the summation and integral signs may be interchanged in the right-hand side of

$$\sum_{n=1}^{\infty} \mu(n)\, f(knx) = \sum_{n=1}^{\infty} \mu(n) \int_0^{\infty} e^{-kxt} \frac{t}{n^2}\, a\left(\frac{t}{n}\right) dt$$

to obtain

$$\int_0^{\infty} e^{-kxt}\, t\, c(t)\, dt.$$

Thus

$$f(x) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-kxt}\, t\, c(t)\, dt = \int_0^{\infty} \frac{t\, c(t)}{e^{xt}-1}\, dt \qquad 0 < x < \infty,$$

the final interchange of integral and summation signs being also justified by (2.10). This completes the proof of the theorem.

3. *Inversion formulas.* We observe first that the Möbius function $\mu(n)$ enables us to express the Laplace integral (2.5) in terms of the Lambert integral (2.1). We prove a companion theorem to Theorem 2.1.

THEOREM 3.1. *If $F(x)$ and $f(x)$ are defined by* (2.1), (2.2), (2.3), (2.5), *then*

$$f(x) = \sum_{n=1}^{\infty} \mu(n)\, F(nx) \qquad 0 < x < \infty, \tag{3.1}$$

the series converging absolutely.

This is the inversion of equation (2.6). It follows immediately from Theorem 2.1 and Lemma 2.3. The absolute convergence of the double series (2.9) is established by use of the relation (2.10).

THEOREM 3.2. *If $F(x)$ is defined by* (2.1), (2.2), (2.3), *and if $a(t)$ is continuous at a point $t_0 > 0$, then*

$$t_0\, a(t_0) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left[\frac{k}{t_0}\right]^{k+1} \sum_{n=1}^{\infty} \mu(n)\, n^k F^{(k)}\left[\frac{nk}{t_0}\right]. \tag{3.2}$$

By formal differentiation of series (2.6) we obtain for every positive integer k

$$F^{(k)}(x) = \sum_{p=1}^{\infty} f^{(k)}(px)\, p^k \qquad 0 < x < \infty. \tag{3.3}$$

The formal step will be justified if the series (3.3) converges uniformly in $c \leqq x < \infty$ for an arbitrary positive number c. Proceeding as in the proof of (2.11) or by use of [9; 182], we see easily that there exists a constant M such that

$$|f^{(k)}(x)| \leqq \frac{M}{x^{k+\delta+1}} \qquad c \leqq x < \infty. \tag{3.4}$$

Hence $$\sum_{p=1}^{\infty} f^{(k)}(px)p^k \ll \sum_{p=1}^{\infty} \frac{Mp^k}{(px)^{k+\delta+1}} \ll \frac{M}{c^{k+\delta+1}} \sum_{p=1}^{\infty} \frac{1}{p^{\delta+1}}.$$

This is sufficient to prove the desired uniform convergence.

Now

$$(3.5) \quad \sum_{n=1}^{\infty} \mu(n)n^k F^{(k)}(nx) = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \mu(n)(np)^k f^{(k)}(npx) = f^{(k)}(x)$$

by Lemma 2.3 provided the double series (3.5) converges absolutely. But in view of (3.4) it is dominated by the convergent double series

$$\frac{1}{x^{k+\delta+1}} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{M}{(np)^{\delta+1}}.$$

Hence (3.5) is established, and the right-hand side of (3.2) is seen to be

$$\lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t_0}\right)^{k+1} f^{(k)}\left(\frac{k}{t_0}\right).$$

But this is the familiar [9; 288] inversion formula for the Laplace transform (2.5) and consequently yields $t_0\, a(t_0)$, since $a(t_0)$ is assumed continuous at t_0.

Theorem 3.3. *Under the conditions of Theorem 3.2*

$$(3.6) \quad t_0 a(t_0) = \sum_{n=1}^{\infty} \mu(n) \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t_0}\right)^{k+1} n^k F^{(k)}\left(\frac{nk}{t_0}\right).$$

Here we have interchanged the symbols for summation and limit in formula (3.2). To justify this we appeal to Theorem 2.2, which states that $F(x)$ may be regarded as a Laplace transform of the function $b(t)$ there defined. By the inversion of such a transform cited above we see that (3.6) becomes

$$(3.7) \quad t_0\, a(t_0) = \sum_{n=1}^{\infty} \mu(n) \frac{t_0}{n^2}\, b\left(\frac{t_0}{n}\right).$$

By the definition of $b(t_0)$ given in Theorem 2.2, equation (3.7) can now be established by another appeal to Lemma 2.3, the double series in question now being

$$\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\mu(n)}{(np)^2}\, a\left(\frac{t}{np}\right).$$

The function $f(x)$ of that lemma is here $x^{-2}a(1/x)$, and the absolute convergence required follows easily from the assumption (2.3).

4. *Operational considerations*. Let us show in detail how formulas (3.2) and (3.6) may be interpreted as $\frac{1}{\Gamma(D)}\frac{1}{\zeta(D)}F(e^{-x})$ and $\frac{1}{\zeta(D)}\frac{1}{\Gamma(D)}F(e^{-x})$, respectively. As we indicated in the introduction the Lambert transform (2.1) may be written as the convolution

$$F(e^{-x}) = \int_{-\infty}^{\infty} G(x-t)\, e^{2t}\, a(e^t)\, dt,$$

where $G(x)$ is the function (5). Using the definition (8), we have

$$\frac{1}{\zeta(D)} F(e^{-x}) = \sum_{n=1}^{\infty} \mu(n)\, F(ne^{-x}),$$

or by Theorem 3.1,

$$\frac{1}{\zeta(D)} F(e^{-x}) = \int_{-\infty}^{\infty} K(x-t)\, e^{2t}\, a(e^t)\, dt \tag{4.1}$$

$$K(t) = e^{-e^{-t}}.$$

Since the right-hand side is a Laplace transform in which a change of variable has been made, we expect its inversion to be effected by the operator $1/\Gamma(D)$. We recall the formal details.

A familiar definition of $\Gamma(x)$ is

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt = \int_{-\infty}^{\infty} e^{-xt} K(t)\, dt.$$

Using the interpretation of e^{-Dt} as a translation through distance $-t$, we see that

$$\Gamma(D)\, e^{2x}\, a(e^x) = \int_{-\infty}^{\infty} K(t)\, e^{2x-2t}\, a(e^{x-t})\, dt,$$

or by (4.1),

$$e^{2x}\, a(e^x) = \frac{1}{\Gamma(D)}\frac{1}{\zeta(D)} F(e^{-x}). \tag{4.2}$$

Of course the actual realization of the operation $1/\Gamma(D)$ is best accomplished by use of one of the familiar inversions of the Laplace transform. It is thus that we have derived formula (3.2).

On the other hand, we saw in Theorem 2.2 that $F(x)$ is the Laplace transform of $tb(t)$. Hence

$$\frac{1}{\Gamma(D)} F(e^{-x}) = e^{2x}\, b(e^x),$$

and

$$\frac{1}{\zeta(D)}\frac{1}{\Gamma(D)} F(e^{-x}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\, e^{2x}\, b\left(\frac{e^x}{n}\right).$$

By Lemma 2.3 the right-hand side of this equation is $e^{2x}\, a(e^x)$.

We have thus permuted the two symbolic operators in (4.2). Since our interpretations of $1/\Gamma(D)$ and $1/\zeta(D)$ involve, respectively, the symbols *lim* and Σ, it is clear why these have been interchanged in (3.2) and (3.6).

In conclusion let us point out why our previous inversion [6] of the convolution (4) is not applicable here. In that general theory we were concerned with an inversion operator $E(D)$ in which the function $E(s)$ was entire. The corresponding function for the Lambert transform is the meromorphic function $[\zeta(s)\,\Gamma(s)]^{-1}$, which has poles at the complex zeros of $\zeta(s)$.

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Harvard University

INTRODUCTION TO A STUDY OF A TYPE OF FUNCTIONAL DIFFERENTIAL AND FUNCTIONAL INTEGRAL EQUATIONS[1]

Lewis Bayard Robinson

Introduction. The following is the first chapter of an exposition of the method of alternating successive approximations. The second chapter has already been printed.[2]

The author was forced to develop a new method because Picard's method of direct successive approximations which has proved to be such a powerful instrument in the field of differential equations and integral equations also required to be somewhat generalized when functional equations were to be investigated. The theory promises to be very extensive.[3]

In a former work the author found a solution of the equation

$$u'(x) = \frac{\lambda}{1-x^2}\, u(x^2) \tag{1}$$

provided that λ is sufficiently small.[4] Here he removes that restriction.

In the following pages the variable x is able to take all the values in the finite complex plane with the exception of the roots of the equation

$$1 - x^{2^r} = 0 \qquad r = 1, 2, 3, \cdots$$

We denote by D the region so defined.

Whenever in this note we use the symbol $e^{2\theta}$ it designates a point on the circumference of the unit circle which we may select arbitrarily except that it must avoid the roots of

$$1 - x^{2^r} = 0.$$

When once selected $e^{i\theta}$ remains constant.

[1]This note was mailed to the Société Mathématique de France April 5, 1940 but of the French text I have only the proof sheets, the war having prevented publication.

[2]See "Revista de Ciencias" No. 456 Ano. XLVIII pages 101-107.

[3]In his researches in this field the author has been greatly encouraged by Professor Hadamard.

[4]See "Revista de Ciencias" Diciembre 1937 pages 139-151.

Part 1

We begin our work by constructing an adjoint to (1).

First let us make the transformation

$$x = 1/\xi .$$

Then

$$u(x) = u(1/\xi) = v(\xi)$$

$$v_\xi'(\xi) = u_x'(x)\frac{-1}{\xi^2} .$$

So we can write (1) and its adjoint thus

$$u_x'(x) = \frac{\lambda}{1-x^2}u(x^2)$$

$$v_\xi'(\xi) = \frac{\lambda}{1-\xi^2}v(\xi^2).$$

Systems of the above type often have singular solutions which are aperiodic when they turn about the origin, which is a point of ramification. However we will not now discuss these singular solutions.

There is a general solution with initial conditions

$$u(0) = u_0 \quad v(0) = v_0 .$$

We write this solution thus

$$u(x) \equiv u_0\{1 + \lambda\, u_1(x) + \lambda^2\, u_2(x) + \lambda^3\, u_3(x) + \cdots\}$$

$$v(\xi) \equiv v_0\{1 + \lambda\, v_1(\xi) + \lambda^2\, v_2(\xi) + \lambda^3\, v_3(\xi) + \cdots\}$$

where

$$u_1(x) \equiv \int_0^x \frac{dt}{1-t^2} \qquad v_1(\xi) \equiv \int_0^\xi \frac{dt}{1-t^2} \equiv \int_\infty^x \frac{dt}{1-t^2}$$

$$u_2(x) \equiv \int_0^x \frac{u_1(t^2)}{1-t^2}dt \qquad v_2(\xi) \equiv \int_0^\xi \frac{v_1(t^2)}{1-t^2}dt \equiv \int_\infty^x \frac{u_1(t^2)}{1-t^2}dt$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$u_n(x) \equiv \int_0^x \frac{u_{n-1}(t^2)}{1-t^2}dt \qquad v_n(\xi) \equiv \int_0^\xi \frac{v_{n-1}(t^2)}{1-t^2}dt \equiv \int_\infty^x \frac{u_{n-1}(t^2)}{1-t^2}dt$$

It is easy to see that we can also write

$$u_1(x) \equiv \int_{e^{i\theta}}^x \frac{1}{1-t^2}dt + a_1 \equiv w_1(x) + a_1$$

$$v_1(\xi) \equiv \int_{e^{-i\theta}}^{\xi} \frac{1}{1-t^2}\,dt + b_1 \equiv \int_{e^{i\theta}}^{x} \frac{1}{1-t^2}\,dt + b_1 \equiv w_1(x) + b_1$$

$$u_n(x) \equiv w_n(x) + a_1 w_{n-1}(x) + \cdots + a_{n-1} w_1(x) + a_n$$

$$v_n(\xi) \equiv w_n(x) + b_1 w_{n-1}(x) + \cdots + b_{n-1} w_1(x) + b_n$$

where

$$w_n(x) \equiv \int_{e^{i\theta}}^{x} \frac{w_{n-1}(t^2)}{1-t^2}\,dt \qquad n = 2,3,4,\ldots$$

$$a_1 = \int_0^{i\theta} \frac{1}{1-t^2}\,dt$$

$$b_1 = \int_0^{-i\theta} \frac{1}{1-t^2}\,dt = \int_{\infty}^{i\theta} \frac{1}{1-t^2}\,dt$$

$$a_n = \int_0^{i\theta} \frac{u_{n-1}(t^2)}{1-t^2}\,dt$$

$$b_n = \int_0^{e^{-i\theta}} \frac{v_{n-1}(t^2)}{1-t^2}\,dt = \int_{\infty}^{e^{i\theta}} \frac{u_{n-1}(t^2)}{1-t^2}\,dt \qquad n = 2,3,4,\ldots$$

Therefore we can write

$$\alpha(\lambda) \equiv u(e^{i\theta}) \equiv u_0\{1 + \lambda a_1 + \lambda^2 a_2 + \cdots\}$$

$$\beta(\lambda) \equiv v(e^{-i\theta}) \equiv v_0\{1 + \lambda b_1 + \lambda^2 b_2 + \cdots\}$$

$$|e^{i\theta}| = 1.$$

Now $u(x)$ converges when λ is arbitrary and

$$|x| \leqq 1$$

x being in the region D. Therefore $\alpha(\lambda)$ and $\beta(\lambda)$ are both integral functions of λ. So we can write the identities

$$\begin{aligned} u(x) &\equiv u_0\{1 + \lambda u_1(x) + \lambda^2 u_2(x) + \lambda^3 u_3(x) + \cdots\} \\ &\equiv \alpha(\lambda)\{1 + \lambda w_1(x) + \lambda^2 w_2(x) + \lambda^3 w_3(x) + \cdots\} \\ &\equiv \alpha(\lambda)\, F(\lambda, x) \end{aligned}$$

$$\begin{aligned} v(\xi) &\equiv v_0\{1 + \lambda v_1(\xi) + \lambda^2 v_2(\xi) + \lambda^3 v_3(\xi) + \cdots\} \\ &\equiv \beta(\lambda)\{1 + \lambda w_1(x) + \lambda^2 w_2(x) + \lambda^3 w_3(x) + \cdots\} \\ &\equiv \beta(\lambda)\, F(\lambda, x). \end{aligned}$$

Part 2

Now we can affirm that

$$\alpha(\lambda)\, F(\lambda,x), \quad \text{also}$$
$$\beta(\lambda)[\alpha(\lambda)\, F(\lambda,x)]$$

converge when λ is arbitrary

$$|x| \leqq 1$$
$$1 - x^{2^r} \neq 0$$
$$\beta(\lambda)\ F(\lambda,x), \quad \text{also}$$
$$\alpha(\lambda)[\beta(\lambda)\, F(\lambda,x)]$$

converge when λ is arbitrary

$$|x| \leqq 1 \qquad |\xi| \leqq 1$$
$$1 - x^{2^r} \neq 0.$$

But we may write

$$U(x) \equiv \beta(\lambda)\ [\alpha(\lambda)\, F(\lambda,x)] \equiv \alpha(\lambda)[\beta(\lambda)\, F(\lambda,x)]\ .$$

Therefore

$$U(x) \equiv \beta(\lambda)[\alpha(\lambda)\, F(\lambda,x)]$$

converges when λ is arbitrary and

$$1 - x^{2^r} \neq 0.$$

and

$$U(x) \equiv \alpha(\lambda)[\beta(\lambda)\, F(\lambda,x)]$$

converges when λ is arbitrary and

$$1 - x^{2^r} \neq 0.$$

The inequalities

$$|x| \leqq 1 \quad \text{or} \quad |x| \geqq 1$$

need no longer be satisfied. In fact x may be any point in the finite complex plane except that it must avoid the roots of

$$1 - x^{2^r} = 0.$$

consider

$$U(x) \equiv \beta(\lambda)\ u(x) \equiv \alpha(\lambda)\ v(\xi)$$

If $u(x)$ and $v(\xi)$ satisfy equation (1), so also does $U(x)$. $\alpha(\lambda)$ and $\beta(\lambda)$ are simply new constants of integration.

Note that we can convert $u(x)$ into a function holomorphic with respect to λ by using only *one* multiplier $\beta(\lambda)$. The variable x may be within or without the unit circle.

Conclusion

The above method is applicable to the equation

$$u'(x) = \frac{P_{n-2}(x)}{Q_n(x)}\, u(x^{\pm p}) + \frac{\mathbf{P}_{n-2}(x)}{\mathbf{Q}_n(x)}$$

where P_{n-2} and $\mathbf{P}_{n-2}$ are polynomials of order $n-2$ and Q_n and $\mathbf{Q}_n$ are polynomials of order n. (This case is studied in Chapter 2.)

The author has been asked if it is possible to select a path of integration always avoiding the roots of

$$1 - x^{2^r} = 0$$

even when

$$|x| = 1.$$

It is possible in an infinity of ways. Rigorous proof is given in another note.[5]

Of course when we have proved convergence when

$$|x| = 1$$

we know, a fortiori, that the series converges when

$$|x| < 1.$$

The question arises. Can we solve our equation by using the method of Fredholm, that is by passing to the limit of an infinite determinant. The answer is Yes. To use this method when x is a real number is not difficult. But suppose that x be a complex number and lies on the vector $\mathbf{w}$.

Suppose that

$$\mathbf{w}^3 = 1$$

Then x^2 will lie on the vector $\mathbf{w}^2$. It is still possible to use the method of Fredholm but the method becomes more complicated.

Suppose we consider the equation

$$u'(e^{\theta}x) = K(x)\, u(e^{2\theta}x^2).$$

[5]See "Revista de Ciencias" Homenaje a La Memoria del Doctor Federico Villareal 1948 pages 141-150. This note contains a theorem on cyclic groups which the author failed to observe until he had read: "Basic Configurations of the Plane Under Certain Groups" by Sr. M. Philip Steele and V. O. McBrien, Mathematics Magazine Sept.-Oct. 1949 p. 5.

Assume that $e^{i\theta}$ is not commensurable with $e^0 = 1$. In this case the author has not been able to use the method of Fredholm.

There is another difficulty. If we use Fredholm's method we must multiply $u(x)$ within the unit circle by $D_1(\lambda)$. Then

$$D_1(\lambda)\, u(x)$$

is holomorphic with respect to λ inside the unit circle. Outside the unit circle we use $D_2(\lambda)$ as multiplier. Then

$$D_2(\lambda)\, u(x)$$

is holomorphic with respect to λ outside the unit circle.

When we use the method of successive approximations we need one multiplier only

$$\beta(\lambda).$$

131 E. North Avenue,
Baltimore 2, Maryland.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

HIGH STANDARDS: SACRED AND PROFANE

Joseph Seidlin

Just as, they say, the devil can quote the Scriptures to his purpose, so, too, the ineffectual, the incompetent, the mediocre, the unimaginative, the lazy, can find shelter behind the plastic curtain of high standards.

It would be both futile and unreasonable to deny the matchless contribution of standards to the growth and development of civilization. It has been said, and should be said over and over again —especially in these trying times for democrary — that standards, — the higher, the better — must be developed, maintained and often defended against mob rule; that the submergence of standards is a prognostic act of a decadent society.

In the history of schooling, often loosely referred to as education, standards have played a major role. We find them in varieties of entrance examinations, achievement examinations, qualifying examinations, eligibility rules, certification licenses, etc., etc. At the higher (and professional) levels of schooling the trend has been "to raise" standards; at the lower levels there either has been no trend or, at least, no agreement among educators as to the nature of the trend.

The two injunctions generally associated with "high standards" are, (1) "keep them out", (2) "put them out". In a democracy and especially in this "age of the common man", it is becoming increasingly difficult to "keep them out". But, *the higher "the standards"*, the greater the "out-put".

The school practices I am about to attack grow out of the illogical ready acceptances of converses. (It cannot all be the fault of teachers of geometry. It must be the so exhausted theory of "transfer of training"). The converse in point is simply this: Since high standards lead to many "failures", *it follows that many failures extablish high standards*. Thus, as I said in the opening sentence, "... the ineffectual, the incompetent, the mediocre, the unimaginative, the lazy can find shelter behind the plastic curtain of high standards".

The following exhibits (case histories) are a mere handful of illustrative material.

Case A. Two students failed in mathematics (not in the same year, and under two different instructors) at University *A*. Each of the students was advised to leave the institution and, even on transfer to another institution, *of lower standards*, to keep away from any work in mathematics. Both of these students transferred to Institution *B*. And, against the proferred advice, each elected first year mathematics. They not only got along, but each developed into a superior student and continued his graduate work in fields closely allied to mathematics (one in physical chemistry and the other in econometrics), obtaining advanced degrees in graduate schools of international renown. What claims for high standards may Institutions *A* and *B* make?

Case B. A student failed calculus at the end of his sophomore year, in an institution where failing calculus is tantamount to being dropped from the institution. However, — and here I quote from the advisor, "if a student insists on taking a reexamination at the end of August and if he does passably well, he may be permitted to go on as a junior on probation"; but this particular student was advised against a re-examination. Fortunately for the student, he tutored with a real teacher of calculus, took the re-examination in August, passed it, and, in due time, became an honor student in mathematics. Somewhere in this case history we have involved high standards.

Case C. A brilliant student of mathematics, having obtained his doctorate, received an appointment as an instructor in a department that zealously guarded its high standards, both sacred and profane. At the end of his first year of teaching the young man was elated at the success he had, especially in his class in calculus. As he announced to me with a great sense of accomplishment, he had more than lived up to the standards of the institution by "flunking" 40% of his class. I remonstrated with him to the effect that if that were his ambition, he might have, with just a little greater effort and ingenuity, flunked 60% of his class.

I recall telling him a story in point: A certain lady came home and elatedly told her husband that she saved $100 by not buying a very fine cloth coat. Her husband, a very thrifty soul, suggested that it would have been a more spectacular saving if she had not bought a mink coat valued at $1000.

In the intervening years, this instructor has become one of the finest teachers in his institution. I am told, incidently, that he very rarely fails a student. What has become of his "high standards"?

Case D. High standards (profane) appear in many guises, even at the secondary school level. In the state of New York, for instance, where often, however mistakenly, a teacher (or a school) is judged by his Regents record, — that is, by the ratio of the number of his pupils who pass the Regents examination to those who take it, — some teachers or-schools eliminate *freely* all pupils who are at all unlikely to pass the examination. Elementary, but it does keep up high standards.

Case E. Even at the professional level, we find this admixture of sacred and profane high standards. School of Medicine *A* admits into its freshman class 100 students; it graduates 70. School of Medicine *B* admits into its freshman class 100; it graduates 100. (In each institution the entering students are painstakingly carefully selected). The administration of School *A* contends that by failing 30% of its entering class it sustains its reputation for high standards. The administration of School *B* claims that it maintains high standards by personal attention, academic nurture, excellent facilities, much and early practical application and, generally, and perhaps most important of all, superior teaching.

Any teacher or administrator not himself a conscious member of "High Standards Anonymous" could readily augment the number and enrich the variety of the above examples. But what purpose will be served by further or greater indictment of an educationally evil practice that is ostensibly based on an apparently educationally sound principle, unless it be a re-evaluation of the principle itself?

In our democracy we are committed to mass schooling at all levels. What moral obligations do institutions of learning and teachers assume in the so often quoted "race between education and catastrophe"? Incontrovertibly it must be to provide increasingly better schooling to greater and greater numbers of educable people. Institutional and professional integrity demands therefore that genuine high standards involve not only "achievement of high order" but *also "for the greatest number of learners"*. Thus, admittedly, standards suffer *either* when quality of achievement is impoverished *or* when mortality is increased. In the unachievable perfect school, none but the mentally and physically crippled could fail.

Teaching — part science, part art, that it is — imposes its own standards. Occasionally there is an apparent clash between high standards in teaching and high standards in the vehicle of teaching, viz, a "teacher" who would not omit the least point in the hyper-rigorous structure of a proof, often omits (ignores) his students. Actually there is no conflict between standards.

Many of our "teachers" are not really teachers. They are mathematicians, physicists, historians, linguists, etc., — not teachers. Many of them are men (and women) of great stature; major contributors to science, technology, and the arts; but they are not teachers. On some scales of worth to humanity they outweigh the teachers; but they are not teachers. They might even be indispensable to institutions of higher learning; but they are not teachers. To them, students are a means; to teachers, students are the end products, — all else is a means. Hence there is but one interpretation of high standards in teaching: standards are highest where the maximum number of students — slow learners and fast learners alike — develop to their maximal capacity.

Let me caution administrators and teachers against another converse. Whereas we may accept the principle that the more nearly perfect the teacher, the fewer the failures in his class; it *does not* follow that the fewer the failures, the more perfect the teacher.

In this era of superabundant "surveys" and "self-surveys" we find one persistent "soul" searching question: "What is the purpose of (1) 'this' institution, (2) 'this' department, (3) 'this' subject, (4) 'this' activity ···?" The *answer,* however arrived at, however phrased — is inevitably the same in meaning; viz., the raison d'etre for the institution, the department, the subject, the activity, is to facilitate learning in *all* areas of human needs and activities. To that end, real, honest, and sincere educators must insist on appropriate (sacred) high standards; must condemn irrelevant (profane) "high standards".

Alfred University.

ELEMENTARY DIFFERENTIAL EQUATIONS

D. H. Hyers

1. *Introduction*

The student of calculus has encountered equations of the form

$$\frac{dy}{dx} = f(x) \tag{1}$$

where one wishes to find y, knowing the derivative of y. This is a very special case of a *differential equation**. This particular type of differential equation can be solved by direct integration. For example, "the solution" of the equation

$$\frac{dy}{dx} = x^2 \tag{2}$$

is $y=\frac{x^3}{3}+C$, where C is an arbitrary constant. Let us note some ways in which Eq. (2) and its solution differ from the solutions of equations studied in elementary algebra. Besides the obvious difference that the operations of calculus enter into the problem, observe that the unknown is a *function* not merely a number. Also, the solution, even as a function of x is not unique, and in fact we obtain an infinite number of solutions of Eq.(2), one for each value of the constant C.

In the language of analytic geometry, in solving Eq.(2), we have solved the problem, "find all curves whose slope at any given abscissa is equal to the square of the abscissa", and the result is a family of parallel cubic curves.

In order to specify a particular curve of the family, an extra condition besides the differential equation must be given. For example, if in addition to having its slope equal to the square of its abscissa, a curve is known to pass through the point (1,1), then the value of C must be $\frac{2}{3}$, and we obtain a unique curve, whose equation $y=\frac{x^3}{3}+\frac{2}{3}$ satisfies both the differential equation (2) and the *initial condition* $y = 1$ when $x=1$. The family of functions $y=\frac{x^3}{3}+C$ is known as the *general solution* of the differential equation (2). In the case just considered, we first found the general solution, and then by appropriate choice of the constant C, obtained the particular solution satisfying the initial condition. This is the classical procedure, but other methods in which the initial conditions enter in from the beginning may also be used.

In general, an *ordinary differential equation* is a relation

*The name arose because equations like (1) can also be written in differential form, $dy = f(x)dx$.

(3) $$F(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2}, \cdots ,\frac{d^ny}{dx^n}) = 0$$

between the independent variable x, the dependent variable y, and one or more of the successive derivatives of y with respect to x. The order n of the highest derivative that occurs is called the *order* of the differential equation. Thus Eq.(1) is a first order equation. A *solution* of the differential equation (3) is a function $y=\varphi(x)$, which satisfies Eq.(3), for all values of x on a certain interval, say $a \leqq x \leqq b$. That is

$$F[x,\varphi(x),\varphi'(x), \cdots \varphi^{(n)}(x)] = 0$$

for all x in the interval*.

The term "ordinary" introduced in the preceding paragraph is used to distinguish these equations from "partial" differential equations, such as

(4) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0,$$

in which there are *two* or more independent variables, and which therefore involve *partial* derivatives. In this brief article we shall confine the discussion to ordinary differential equations.

2. *The physical origin of differential equations.*

As soon as the calculus was invented, it became possible to formulate many hitherto unsolved physical problems by means of differential equations. For example, in problems concerning the dynamics of a particle, we usually are given the forces acting on the particle, and wish to determine the motion. According to Newton's second law of motion, force = mass times acceleration. For the case of the force and the motion both in the same fixed direction, one obtains just one differential equation

(5) $$m \cdot \frac{d^2x}{dt^2} = F,$$

where x represents the displacement of the particle in the direction of the force measured from some fixed point, m is the mass of the particle, F represents the resultant force on the particle at the time t, and may be a function of t, x, and $\frac{dx}{dt}$.

The most satisfactory way of discussing the theory of electrical circuits is also by means of differential equations, at least when the problem concerns variable currents, as in radio engineering, etc. As a simple example, consider an electrical circuit consisting of an inductance (L henrys) and a resistance (R ohms) in series, driven by a

*$\varphi^{(k)}(x)$ denotes the kth derivative of the function $\varphi(x)$. The function may be defined implicitly also. (See section 3).

generator (or battery) which impresses an electromotive force E on the circuit (see fig. 1). Here it is assumed that L and R are constants, while E may be a constant (in case of a battery) or, in general, some known function of the time t.

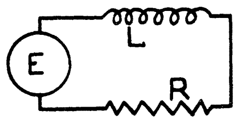


Fig. 1

According to the fundamental laws of circuit theory, we can arrive at the equation of the circuit by equating the sum of the potential drops over each portion of the circuit to the impressed electromotive force E. This gives

$$L\frac{dI}{dt}+RI = E, \tag{6}$$

a first order differential equation, in which the current I is the dependent variable, and the time t the independent variable.

A third problem, which one will find in many elementary texts on differential equations, concerns the mixture of fluids. This type of problem occurs in chemical engineering. To obtain a simple differential equation here, it is necessary to make certain "simplifying assumptions" in the physical problem, which are actually only approximately true.

For example, suppose a vat contains initially 1000 gals. of salt brine, consisting of 200 lb. of salt dissolved in water. If pure water is running in at the rate of 3 gals. per min., and the solution (after being "perfectly" mixed) runs out at the same rate, the problem is to find the concentration of salt (lb. per gal.) at any time t (min.) after the process starts. The principal simplifying assumption* is that of perfect mixing, which enables us to consider the concentration uniform throughout the solution in the vat at any instant. This is approximately true when the volume of liquid is large relative to the rates of inflow and outflow, and when the solution is kept agitated sufficiently.

To set up the differential equation, let s denote the number of lb. of salt in the vat at time t, and consider the change Δs in this amount during the time interval from t to $t+\Delta t$. Now the concentration of salt in the vat at time t in lb. per gal. is $C=s/1000$, since the volume of solution always remains constant and equal to 1000 gals. In the time interval Δt, no salt flows in, and the amount that flows out is approximately $3C\Delta t$, if Δt is small. Hence we have, approximately:

$$\Delta s = -\frac{3s}{1000}\Delta t$$

Dividing by Δt, and letting Δt tend to zero, we have in the limit

*We are also assuming that the amount of salt in the vat varies continuously, which is true "macroscopically" but not "microscopically". See "Differential Equations" by R. P. Agnew, pp. 46-52 for a discussion of this point.

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = -.003\, s, \tag{7}$$

which is the differential equation for the problem. To this we must add the initial condition:

$$s = 100 \text{ lb.} \quad \text{when } t = 0. \tag{8}$$

To solve equation (7), first write it in differential form $ds = -.003\, s\, dt$, and then divide by s:

$$\frac{ds}{s} = -.003\, dt$$

It can now be integrated, giving*

$$\ln s = -.003\, t + A,$$

where A is the arbitrary constant. But from (8) it follows that $A = \ln 200$. Hence $\ln (s/200) = -.003\, t$, or

$$s = 200\, e^{-.003 t}. \tag{9}$$

Thus the amount of salt in the solution at any time t is given by Eq.(9).

Of course the mixture problem as well as the other types of problems mentioned in this section may be varied a great deal. This section is intended only to give a few simple examples and it is impossible here to illustrate fully the enormous importance of differential equations in physics, chemistry and engineering. Most of the fundamental physical laws can best be stated by means of differential equations, either ordinary or partial, and some of the more complicated laws could hardly be stated at all without them**.

3. *Elementary Methods of Solution of Certain First Order Equations.*

Almost simultaneously with the discovery of the calculus by Newton and Leibnitz in the 17th century, mathematicians of the time began developing methods for the solution of differential equations. The object of most of these investigations was to express the solution of a given differential equation in terms of a finite number of known functions, and the methods developed by the Bernoullis, by Euler and others over a hundred year period beginning about 1690 still form the basis of our elementary college courses in the subject. At first it was hoped that every differential equation could be solved in this way, but it gradually became evident that this was a hopeless task. We cannot even solve every *first order* equation in this way. However, we shall

*ln s stands for the natural logarithm, $\log_e s$. This notation will be used throughout the article.

**For example, Maxwell's laws of electromagnetic wave propagation.

consider one or two simple types of differential equations in which it is possible, by elementary methods, to find the general solution in "closed form" or at least to reduce the problem to a problem in calculating integrals.

We consider first order differential equations which may be written in the form $M + N\,dy/dx = 0$ or in the differential form

$$M\,dx + N\,dy = 0, \tag{10}$$

where M and N are functions* of both x and y.

Separable Equations. In the special case where M and N are each factorable into a function of x alone and a function of y alone, that is

$$M = f(x) \cdot \varphi(y); \quad N = g(x) \cdot \psi(y), \tag{11}$$

it is an easy matter to solve Eq.(10) by the method of separating the variables.

We have already employed this method, in solving Eq.(7). As a further illustration, consider the equation

$$(x+1)\,y\,dx + x\,(y+1)\,dy = 0 \tag{12}$$

Dividing** by xy and transposing gives

$$\frac{y+1}{y}\,dy = -\frac{x+1}{x}\,dx, \tag{13}$$

an equation in which the variables are "separated", that is, all the x's are on one side of the equation, and all the y's are on the other. We are looking for a relation between y and x, free of differentials. Whatever the functional relation between x and y, (we may consider y as a function of x, but this function is unknown at present, namely the one we are trying to find) we can now integrate the equation, obtaining

$$y + \ln y + x + \ln x = C, \tag{14}$$

where C is a constant of integration. The last equation is the general solution. Observe that even in such a simple case the solution (14) defines y only *implicitly* as a function of x.

In case the functions M and N are not of the special form (11) then the variables cannot be separated. However there are some cases in which a properly chosen substitution or change of variables may be used to transform the equation into a separable equation. For example the equation

$$(x^2 + y^2)\,dx + x\,y\,dy = 0$$

*It is assumed that M and N are continuous in the pair (x, y) and that neither function is identically zero.

**At this step it is necessary to assume that $xy \neq 0$. As a matter of fact the origin is a so called "singular point" for this differential equation.

is not separable as it stands, but if we put $y = v\,x$, where v is a new dependent variable, replacing y, we have $dy = v\,dx + x\,dv$, and the equation becomes $(x^2 + x^2v^2)\,dx + x^2\,v\,(v\,dx + x\,dv) = 0$. After dividing by x^2 and collecting terms, this may be written

$$(1 + 2v^2)\,dx + x\,v\,dv = 0,$$

in which the variables may be separated. The method is successful because the x^2 could be divided out. This same substitution works on any first order equation in which both M and N are homogeneous polynomials of the same degree (i.e. every term in M and in N is of the same degree in the variables x, y). Such equations are a special case of the equations called "homogeneous" in the textbooks*.

At this point something should be said about the *logic* involved in finding solutions, a topic not always discussed in elementary textbooks on the subject. Actually (as in algebra) in "solving" the above examples, we have *assumed the existence* of a solution at the first step, and then tried to find it. To complete the argument logically we should substitute the supposed solution into the original differential equation, and verify that it *actually is a solution*, or else assure ourselves by some other means that this is the case. To illustrate, let us return to Eq.(12). The logical steps run as follows. If Eq.(12) has a solution y then we can carry out the necessary steps to form in succession Eqs.(13), (14). The relation (14) is then "a candidate" for the solution. We can now prove that this function is actually a solution, for each value of the constant C, by simply reversing the steps, and writing successively Eqs.(14), (13) and (12), or by substituting the function directly into Eq.(12).

In case the reader thinks this is splitting hairs, let him consider the differential equation

$$\sqrt{\frac{dy}{dx}} + x^2 = 0, \tag{15}$$

where the radical, as usual, indicates the positive square root of the derivative. The function $y = x^5/5 + C$ which is obtained by transposing x^2, squaring and integrating, does not satisfy the equation. In fact the equation has no real solution.

4. *Geometrical Interpretation of First Order Equations*

A differential equation of the form

$$\frac{dy}{dx} = f(x,y) \tag{16}$$

gives us the slope of the solution curve $y = \varphi(x)$ at an arbitrary point (x,y). It is possible to exploit this simple fact in making an approximate plot of the solution curves, whether or not the equation can be

*In the general homogeneous equation of the first order, M and N are required only to be homogeneous *functions* of the same degree, and not necessarily polynomials. The term "homogeneous" is unfortunately used in a different sense in connection with linear equations. See for example, Agnew, loc. cit., p. 85.

solved by elementary methods. As an illustration consider the equation

$$\frac{dy}{dx} = xy + 1. \tag{17}$$

and let us see how much we can tell about the solution curves without actually carrying out the solution. We observe first that if a solution curve has a (relative) maximum or minimum point, it must occur when the right member of Eq.(17) vanishes, that is, at some point of the hyperbola whose equation is

$$xy = -1 \tag{18}$$

Next, if we differentiate Eq.(17), we get

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y = x^2y + x + y. \tag{19}$$

From Eq.(19) it follows that d^2y/dx^2 is positive when x and y are both positive, so that the solution curves are concave upwards in the first quadrant. Similarly, they are concave downwards in the third quadrant. By substituting Eq.(18) into Eq.(19) we can see that at a point at which dy/dx is zero, the second derivative is equal to y. Hence the upper branch of the hyperbola (18) consists of *minimum* points for the solution curves, while the lower branch consists of *maximum* points. To study the concavity further, we may set the left member of Eq.(19) to zero, and solve for y. This gives us the curve whose equation is

$$y = -\frac{x}{x^2 + 1} \tag{20}$$

as the locus of inflection points (points where d^2y/dx^2 changes sign).

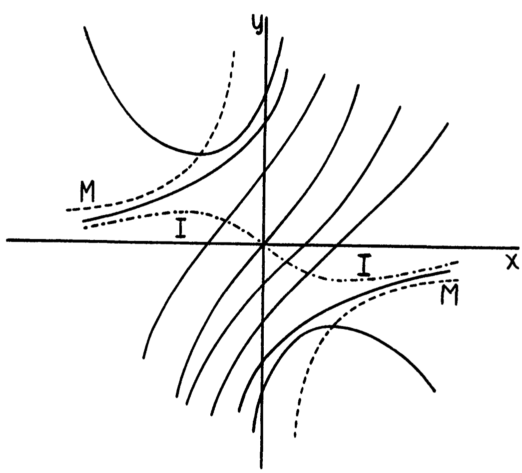


Fig. 2

- - - - - Locus of maxima and minima (*m*)
· - · - · Locus of inflection points (*i*)
———— Solution curves

At points above the curve (20), the solution curves are all concave upwards, and at points below this curve, they are concave downwards. Putting all these facts together allows us to picture the general form of the family of solution curves, as in fig. 2.

A more systematic method, which may be used alone or in combination with the above, is the *method of isoclines*. The isoclines for Eq.(16) are simply the curves of the family

$$f(x,y) = C, \tag{21}$$

obtained by placing the right member of (16) equal to a constant. The reason for the name is evident, for all the solution curves crossing a given isocline are "equally inclined" to it, since at every point of crossing they have the same slope C. By plotting enough curves of this family, and drawing short segments of slope C at intervals along the isocline corresponding to a given value of C, we can often see the trend of the solution curves, and in fact, obtain first approximations to the solutions.

5. *Linear Equations.*

An equation is called linear* in y if it is of the first degree in the dependent variable y and its derivatives. Thus $2xy'' + y' + xy = x^2$ and $xy' + y = 1$ are linear equations, where primes denote derivatives. The form of the general linear equations of orders one and two are:

$$a_0(x)\frac{dy}{dx} + a_1(x)\,y = b(x)$$

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = b(x)$$

The functions $a_0(x)$, $a_1(x)$, $a_2(x)$ are called the *coefficients* of the equation. In case $b(x) \equiv 0$, the equation is said to be homogeneous and linear.

An important characteristic of homogeneous linear equations, sometimes called the "principle of superposition" is that the sum of two solutions is itself a solution, and any constant times a solution is also a solution. In other words, if y_1 and y_2 are solutions, so is $z = c_1y_1 + c_2y_2$, as is readily verified by substituting z into any linear homogeneous equation, of which y_1 and y_2 are solutions. This principle makes the theory and the actual solution of linear differential equations much easier than in the case of non-linear equations.

We have seen in the introduction that the "general solution" of a differential equation of the first order contains one "arbitrary

*The term linear is here used in its algebraic sense, meaning "of the first degree", and comes from the fact that in plane analytics, every straight line has an equation of the first degree.

constant", arising from a single integration process. In a second order differential equation, we will have to integrate twice, and so would expect two arbitrary constants to occur, and so on for any order. In general, these constants may come into the general solution in a very complicated way, but in linear equations, they always occur linearly, as can be shown by means of the principle of superposition. This will be illustrated in the examples to follow.

In the case the coefficients of the linear differential equation are constants, the equations are quite easy to solve. Consider first the homogeneous case and take a first order equation

$$\frac{dy}{dx} + k y = 0 \tag{22}$$

Here the variable may be separated, and we obtain

$$\frac{dy}{y} = -k\,dx$$

$$\ln y = -k x + A, \quad \text{or}$$

$$y = e^{-k x + A}$$

Putting $e^A = C$, this may be written $y = Ce^{-kx}$, where C is an arbitrary constant, replacing A. It can be verified that $y = Ce^{-kx}$ is a solution of (22) for every real value of C, positive, negative, or zero, by direct substitution.

From the fact that the derivative of the exponential function is equal to the function itself, which is the basic reason why it appeared in the solution of (22), we can readily obtain solutions of higher order homogeneous linear equations. For example, to solve the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = 0, \tag{23}$$

we try a solution of the form $y = e^{mx}$, and substitute into (23). Since $\frac{dy}{dx} = m\,e^{mx}$ and $\frac{d^2y}{dx^2} = m^2 e^{mx}$, we have $m^2 e^{mx} - 4m\,e^{mx} - 5\,e^{mx} = 0$, or, since e^{mx} cannot be zero,

$$m^2 - 4m - 5 = 0 \tag{24}$$

Thus $m = 5$ or $m = -1$. Therefore $y_1 = e^{5x}$ and $y_2 = e^{-x}$ are solutions, which again can be readily verified by substitution. By the principle of superposition, the combination

$$y = c_1y_1 + c_2y_2 = c_1e^{5x} + c_2e^{-x} \tag{25}$$

is also a solution for every choice of c_1 and c_2. Equation (25) is the *general solution* of (23), since it contains two arbitrary constants which can be chosen to satisfy any given initial values of y and y'.

The solution of the corresponding homogeneous equation is also useful in the non-homogeneous case. If, by any method, we can obtain one single solution (called a particular integral) of the non-homogeneous equation, its general solution is obtained by adding this particular integral to the general solution of the corresponding homogeneous equation. This is another consequence of the principle of superposition, and can easily be verified by substitution.

To illustrate, consider the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = 10x + 3 \tag{26}$$

To find a particular integral, we may "guess" at a solution of the form $y_p = ax + b$ (this is sensible, since by inspection, the left member of (26) applied to a linear function yields a linear function) and substitute in (26):

$$0 - 4a - 5ax - 5b \equiv 10x + 3 \quad \text{(for all } x\text{)}$$

$$-5ax - 5b - 4a \equiv 10x + 3$$

On equating coefficients of x and the constant terms we have $-5a = 10$, $-5b - 4a = 3$. Hence $a = -2$, $b = 1$. Our particular solution is $y_p = -2x + 1$. The homogeneous equation corresponding to (26) is (23), which has already been solved. Thus the general solution of (26), which is the sum of the general solution of (23) and the particular integral we have found for (26) is

$$y = c_1 e^{5x} + c_2 e^{-x} - 2x + 1.$$

In the case of homogeneous linear equations with constant coefficients, the substitution $y = e^{mx}$, where m is a constant to be determined, will always result in an algebraic equation in m whose degree is equal to the order of the equation. In case the roots of this algebraic equation are all real and distinct, the solution can be written down at once as a sum of exponential functions, as soon as one knows these roots. However, if some of the roots coincide, the solution is no longer a sum of simple exponential functions, but contains terms of the form $x^p e^{mx}$ as well. Finally in case of complex roots, a suitable interpretation of exponential functions with imaginary arguments must be given, which Euler did by means of his famous formula $e^{ix} = \cos x + i \sin x$. This leads to terms of the form $e^{ax} \cos bx$ and $e^{ax} \sin bx$, where $m = a + ib$. ($i = \sqrt{-1}$). To summarize, every homogeneous linear equation with constant coefficients has a general solution consisting of a sum of terms of at most the three types $cx^p e^{mx}$, $cx^p e^{ax} \cos bx$, $cx^p e^{ax} \sin bx$, where p is zero or a positive integer.

We conclude with a physical example. Consider the motion of a mass M hanging on a spring. If L_0 is the length of the spring in equilibrium under the weight and x is the displacement of this mass from the equilibrium position, then the length L at a given time t is $L = L_0 + x$. The differential equation of motion may be obtained from (5). If frictional forces are neglected, and we consider only displacements which are small enough to be directly proportional to the force, then we may take $F = -kx$ and (5) becomes $M\,d^2x/dt^2 = -kx$, or

$$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0, \tag{27}$$

which is a homogeneous linear equation of the second order, with constant coefficients. Using the procedure described above, put $x = e^{mt}$ in (27) and obtain the algebraic equation $m^2 + \omega^2 = 0$, where $\omega = \sqrt{k/M}$. The solutions $m = \pm\omega i$ of this algebraic equation being imaginary, it is necessary to eliminate the resulting exponentials $e^{i\omega t}$ and $e^{-i\omega t}$ by means of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. The general solution of (27) is then

$$x = A\cos\omega t + B\sin\omega t. \tag{28}$$

To obtain the displacement as a definite function of the time t we must know two additional conditions to determine the constants A and B. Suppose for example that the motion is started by pulling the mass down from the equilibrium position to a displacement x_0 and then releasing it (from rest). The initial conditions are then $x = x_0$ and $v = dx/dt = 0$ for $t = 0$. By substituting the first of these conditions into (28) we have $A = x_0$, and by first differentiating (28) and then using the condition on the derivative at $t = 0$ we see that $B = 0$. The solution then becomes $x = x_0\cos\omega t$. Thus the motion is a vibration of frequency $\omega/2\pi$ and the maximum displacement is x_0, as may be seen by plotting x against t. Theoretically, the vibration would be maintained with the same amplitude x_0 indefinitely. This results from the assumption of zero frictional forces.

6. *Conclusion*

The reader will have obtained from this article but a brief glance at the bulging bag of tricks devised by the mathematicians of the 18th century for the solution of special differential equations. For more samples he is referred to the textbooks on elementary differential equations. In perusing such a book he will observe a trace of theory perhaps in the chapter on linear equations and a hint of an important method in the chapter on solutions by series. To progress much beyond this elementary stage, it is necessary to sharpen up powerful tools of analysis such as power series, contour integration, Fourier series, topology, and apply them not only in trying to obtain specific solutions, but in trying to answer more general questions. For example we may ask whether a given type of differential equation always has a solution, and if so is it the only one satisfying various types of conditions. Next, what general properties does the solution have, and how many of these can be found without determining the solution explicitly (since this may be a hopeless and unrewarding task)? Many such questions have been answered by the researches of the last one hundred years, but many more remain to be answered. As to the solution of specific intractable differential equations, this is often accomplished today by means of differential analysers or other mathematical machines.

University of Southern California.

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known text-books should not be submitted.

In order to facilitate their consideration, solutions should be submitted on separate, signed sheets within three months after publication of the problems.

All manuscripts should be typewritten on 8½″ by 11″ paper, double-spaced and with margins at least one inch wide. Figures should be drawn in india ink and in exact size for reproduction.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, Calif.*

PROPOSALS

56. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Find a nine-digit integer of the form $a_1a_2a_3b_1b_2b_3a_1a_2a_3$ which is the product of the squares of four distinct primes, $a_1 \neq 0$, $b_1b_2b_3 = 2(a_1a_2a_3)$.

57. *Proposed by J. S. Cromelin, Clearing Industrial District, Chicago, Ill.*

The floor of a room twice as long as wide was laid by Art, Jake, Pete and Henry with 6″ by 6″ tile. Henry noticed that the length of the room in feet equalled his age. The boss, who had 36 dollar-bills and 40 dimes, paid off the men at the rate of two cents per tile. Art got $2.00 more than Jake, Pete got 40¢ more than Art, and each was paid just what he earned. Henry was given $5.00 less than Jake, but had to give back some change. All this happened on Henry's birthday. How old was he?

58. *Proposed by W. B. Clarke, San Jose, California.*

Through a point P in the plane of a given triangle lines are drawn bisecting the area of the triangle. Discuss the location of points P for which there are one, two, or three bisecting lines.

59. *Proposed by D. L. MacKay, Manchester Depot, Vermont.*

The definition of regular polyhedrons gives three requirements: (a) faces regular polygons, (b) faces congruent, (c) polyhedral angles congruent. Give illustrations of polyhedrons possessing each pair of these requirements but not the third.

60. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

The tangents MA, MB from a point M to a parabola (P) meet the curve in the distinct points A, B. Find the locus of the point M, if the circle MAB passes through the vertex of (P).

61. *Proposed by R. E. Horton, Los Angeles City College.*

Find the value of the determinant D in which $a = \sin\theta$ and $b = \cos\theta$.

$$D = \begin{vmatrix} a^3 & a^2b & ab & b^2 & ab & b^2 \\ a^2b & a^3 & b^2 & ab & b^2 & ab \\ ab & b^2 & a & b & a & b \\ b^2 & ab & b & a & b & a \\ ab & b^2 & a & b & 0 & 0 \\ b^2 & ab & b & a & 0 & 0 \end{vmatrix}$$

62. *Proposed by E. P. Starke, Rutgers University.*

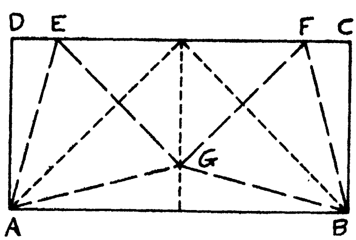
A right cylinder whose cross-section is an ellipse of eccentricity e is placed with its axis horizontal upon an inclined plane, and it does not roll down. What is the greatest possible inclination of the plane? Assume there is no resistance to rolling but that the cylinder cannot slide down the plane.

SOLUTIONS

Five Points on a Rectangle

37. [Sept. 1949] *Proposed by Leo Moser, Winnipeg, Canada.*

Given 5 points in or on a 2×1 rectangle. Show that the smallest distance determined between any 2 of them is $\leqq 2\sqrt{2-\sqrt{3}}$ and that this is the largest number for which the result is true. (Suggested by the game of Russian Billiards.)



Solution by Dewey Duncan, East Los Angeles Junior College. Being ignorant of the rules of Russian Billiards, I am confused by the statement of the problem. 5 points may be placed in coincidence anywhere on the rectangle, yielding the smallest possible distance between any two of them as $0 \leqq 2\sqrt{2-\sqrt{3}}$. Q.E.D. However, this is not the largest number of points for which the result is true, since any number of points would yield the same result.

Perhaps the intent of the problem is to place the 5 points so that they may be most widely separated, or rather, so that the distance between any two of them should be "a maximum". In this event, on the rectangle $ABCD$, with $AB = CD = 2$ and $BC = DA = 1$, we place a point on A, a point on B, two points E, F on CD so that $CF = DE$, and the fifth point G on the perpendicular bisector of AB. If the maximum separation is to be achieved, then AEG and BFG must be equilateral triangles. The problem thus reduces to one of inscribing an equilateral triangle in a unit square, one vertex coinciding with a vertex of the square. This is done by constructing two 30° angles on a diagonal of the square and joining the intersections of their sides with the sides of the square. Hence, the side of the triangle is equal to $\sec 15^\circ$ or $2\sqrt{2-\sqrt{3}}$. Inspection of this arrangement shows that movement of any one of the 5 points must diminish the distance between it and at least one of the remaining points to a value $< 2\sqrt{2-\sqrt{3}}$.

Five Intersecting Great Circles

38. [Sept. 1949] *Proposed by Leo Moser, Winnipeg, Canada.*

Show that 5 or more great circles on a sphere, no 3 of which are concurrent, determine at least one spherical polygon having 5 or more sides.

Solution by Dewey Duncan, East Los Angeles Junior College. Three great circles intersecting in distinct points divide a spherical surface into 8 spherical triangles occurring in symmetric pairs which lie in opposite hemispheres. Let 2 great circles intersect at O and O', and let them meet a 3rd great circle in points A, A' and B, B', respectively. On the hemisphere bounded by the great circle through A, B, A', B' and containing point O, examine the configuration obtained by drawing a 4th great circle. This 4th great circle meets the 3rd great circle at the diametrically opposite points C, C', the semicircle AOA' at P, and the semicircle BOB' at Q. Thus a triangle OPQ is formed and is surrounded cyclically by triangle AOB, quadrilateral $BOPC$, triangle CPA', quadrilateral $A'PQB'$, triangle $B'QC'$, and quadrilateral $C'QOA$. Hence 4 great circles having all intersection points distinct must always yield 8 spherical triangles and 6 spherical quadrilaterals on a sphere.

Consider again the hemisphere bounded by the original 3rd great circle and containing the aforesaid configuration. Let the 5th great circle cut the 3rd great circle in points D and D', one of which must lie on a side of a triangle, the other on a side of a quadrilateral. Let D lie on side AB of AOB and D' lie on side $A'B'$ of $A'PQB'$. Observe that the sides of the triangle OPQ are also sides of quadrilaterals. Now DD' *will* or *will not* enter triangle OPQ.

If DD' does enter triangle OPQ it must cut 2 of its sides, thereby dividing triangle OPQ into a triangle and a quadrilateral. Semicircle DD' must also divide AOB into a triangle and a quadrilateral, the quadrilateral from which it enters OPQ into a triangle and a pentagon, and $A'PQB'$ into 2 quadrilaterals. Hence the 5th circle yields a configuration on the hemisphere consisting of 5 triangles, 5 quadrilaterals and one pentagon.

If DD' does not enter OPQ it must pass through four consecutive peripheral polygons, thereby dividing AOB into a triangle and a quadrilateral, the adjacent quadrilateral into 2 quadrilaterals, the next adjacent triangle into a triangle and a quadrilateral, and $A'PQB'$ into a triangle and a pentagon. Again, we have 5 triangles, 5 quadrilaterals and one pentagon.

Hence, the configuration formed on a sphere by 5 great circles, no 3 of which are concurrent, consists of *exactly* 10 triangles, 10 quadrilaterals and 2 pentagons. Furthermore, a 6th great circle will either leave the pentagon intact, will divide it into a pentagon and a quadrilateral, or will divide it into a triangle and a hexagon. Hence, if on the sphere there are n great circles ($n \geq 5$), no 3 of which are concurrent, there will always be at least one spherical polygon with 5 or more sides.

Determinant Equal to Sum of Reciprocals of Integers

39. [Sept. 1949] *Proposed by Leo Moser, Winnipeg, Canada.*

Let $A_{ij} = [(i+1)^{j+1} - 1]/(i+1)!$, $i = 1,2,3, \cdots, n$, $j = 1,2,3, \cdots, n$. Show that the nth order determinant

$$|A_{ij}| = 1 + 1/2 + 1/3 + \cdots + 1/(n+1).$$

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. If we take common factors from the rows and perform the operations $\mathrm{col}_k - \mathrm{col}_{k-1}$, $k = 2,3, \cdots, n$, we have

$$|A_{ij}| = \frac{n!}{2!3!\cdots(n+1)!}\begin{vmatrix} 2+1 & 2^2 & 2^3 & \cdots & 2^n \\ 3+1 & 3^2 & 3^3 & \cdots & 3^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (n+1)+1 & (n+1)^2 & (n+1)^3 & \cdots & (n+1)^n \end{vmatrix}$$

$$= \frac{n!(n+1)!}{2!3!\cdots(n+1)!}\begin{vmatrix} 1 & 2 & 2^2 & \cdots & 2^{n-1} \\ 1 & 3 & 3^2 & \cdots & 3^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & n+1 & (n+1)^2 & \cdots & (n+1)^{n-1} \end{vmatrix}$$

$$+ \frac{n!}{2!3!\cdots(n+1)!}\begin{vmatrix} 1 & 2^2 & 2^3 & \cdots & 2^n \\ 1 & 3^2 & 3^3 & \cdots & 3^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & (n+1)^2 & (n+1)^3 & \cdots & (n+1)^n \end{vmatrix}$$

The last two determinants are alternants [Muir and Metzler, *A Treatise on the Theory of Determinants*, Longmans, Green and Co. (1933), pages 321-363]. The first is expressible as a difference-product which obviously is $2!3!\cdots(n-1)!$, the second as the same difference-product multiplied by a symmetric function of degree $n-1$. On reducing, we have

$$|A_{ij}| = 1 + [3\cdot4\cdot5\cdots(n+1) + 2\cdot4\cdot5\cdots(n+1) + \cdots + 2\cdot3\cdot4\cdots n]/(n+1)!$$

$$= \sum_{k=1}^{n+1} 1/k.$$

Also solved by B. K. Gold, Los Angeles City College.

For other problems involving this sum, see *American Mathematical Monthly*, **41**, 48, (1934); **56**, 109, 417, (1949); and *School Science and Mathematics*, **47**, 836, (1947).

Syzygies of Sums of Powers of Consecutive Integers

40. [Sept. 1949] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Let x be a positive integer and $S_n = 1 + 2^n + 3^n + \cdots + x^n$. Prove the following Pythagorean relations:

a) $(48S_7 + 160S_9 + 48S_{11} + 1)^2 = (8S_3 + 24S_5)^2 + [(4S_3 + 12S_5)^2 - 1]^2$.

b) $(64S_9 + 448S_{11} + 448S_{13} + 64S_{15} + 25)^2$

$$= (160S_5 + 160S_7)^2 + [(16S_5 + 16S_7)^2 - 25]^2.$$

Solution by J. M. Howell, Los Angeles City College. A well-known formula for the sum of the nth powers of the first x consecutive integers is

$$S_n = x^{n+1}/(n+1) + x^n/2 + B_1 n x^{n-1}/2! - B_2 n(n-1)(n-2)x^{n-3}/4! + \cdots,$$

where the B_i are Bernoulli numbers and the term containing x^0 and those that follow are omitted. From this formula we obtain the following specific values of S:

$$S_1 = x(x+1)/2; \quad S_3 = x^2(x+1)^2/4; \quad S_5 = x^2(2x^4 + 6x^3 + 5x^2 - 1)/12;$$
$$S_7 = x^2(3x^6 + 12x^5 + 14x^4 - 7x^2 + 2)/24;$$
$$S_9 = x^2(2x^8 + 10x^7 + 15x^6 - 14x^4 + 10x^2 - 3)/20;$$
$$S_{11} = x^2(2x^{10} + 12x^9 + 22x^8 - 33x^6 + 44x^4 - 33x^2 + 10)/24;$$
$$S_{13} = x^2(30x^{12} + 210x^{11} + 455x^{10} - 1001x^8 + 2145x^6 - 3003x^4 + 2275x^2 - 691)/420;$$
$$S_{15} = x^2(3x^{14} + 24x^{13} + 60x^{12} - 182x^{10} + 572x^8 - 1287x^6 + 1820x^4 - 1382x^2 + 420)/48.$$

Since the right-hand member of a) is of the form $(2y)^2 + (y^2 - 1)^2 = (y^2 + 1)^2$, and the S_i are positive, it is sufficient to show that $48S_7 + 160S_9 + 48S_{11} = (4S_3 + 12S_5)^2$. Upon substituting the values of the S_i established above, each member of the last equation reduces to $4n^6(n+1)^6$, thus establishing the identity a).

Similarly, the right-hand member of b) is of the form $(10y)^2 + (y^2 - 25)^2 = (y^2 + 25)^2$. Therefore, to establish b) it is sufficient to show that $64S_9 + 448S_{11} + 448S_{13} + 64S_{15} = (16S_5 + 16S_7)^2$. Now each member of the last equation reduces to $4n^8(n+1)^8$, which completes the proof.

Also solved by the proposer, who has developed a number of other syzygies involving the S_i. Some of those recorded in his privately printed paper "Bernoulli's Asterisks" are $6S_1S_2 = S_2 + 5S_4$,

$3(S_2)^2 = S_3 + 2S_5$, $15(S_4)^2 = 6S_9 + 10S_7 - S_5$, $21(S_6)^2 = 6S_{13} + 21S_{11} - 7S_9 + S_7$, and $(4S_3 + S_1)^2 = (4S_3)^2 + (3S_2)^2$.

Some other identities which have been established are $S_3 = (S_1)^2$ [*School Science and Mathematics*, **14**, 722, (1914)]; $S_5 + S_7 = 2(S_3)^2 = 2(S_1)^4$ [*American Mathematical Monthly*, **22**, 99, (1915); **43**, 105, (1936); *School Science and Mathematics*, **31**, 346, (1931); **45**, 775, (1945)]; and $S_9 + 7S_{11} + 7S_{13} + S_{15} = 16(S_1)^8$ [*School Science and Mathematics*, **34**, 312, (1934)]. If to these we add $S_3 + 3S_5 = 4(S_1)^3$ and $3S_7 + 10S_9 + 3S_{11} = 16(S_1)^6$, and appropriately substitute in a) and b), each reduces to an identity in S_1.

An Archaic Age Problem

41. [Sept. 1949] *Question by R. L. Krueger, Wittenburg College.*

I wonder if you can tell me anything about the following problem. It was sent to me by a former student and we do not know the original source, nor can we seem to interpret it correctly. "3 children: one has lived a diminished evenly even number of years, another a number also diminished, but evenly uneven, while a third, an augmented number unevenly even. What are the ages of the children?"

Solution by Dewey Duncan, East Los Angeles Junior College. These types of numbers occur in Book VII of Euclid's Elements. [T. L. Heath, *The Thirteen Books of Euclid's Elements*, 2nd Edition (1926), Volume II, pages 281 - 4.] In effect, if m and n are positive integers, then 2^n is an evenly even number, $2(2m+1)$ is an evenly uneven number, and $2^{n+1}(2m+1)$ is an unevenly even number. Moreover, in the *Encyclopedia Britannica*, 14th Edition, page 343 under the caption AGE, childhood is defined as extending from the 7th year to the 14th year of human life.

The diminished evenly even numbers, $2^n - 1$, constitute the series, 1, 3, 7, 15, $\cdots$. The diminished evenly uneven numbers, $2(2m+1) - 1$, form the series, 5, 9, 13, 17, $\cdots$. The two smallest augmented unevenly even numbers, $2^{n+1}(2m+1) + 1$ are 13 and 21. Hence the ages of the children are 7, 13 and 13 or 7, 9 and 13, which latter solution would be unique under the restriction that all the ages are different.

QUICKIES

From time to time as space permits this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 1. After a typist had written ten letters and had addressed the ten corresponding envelopes, a careless mailing-clerk inserted the letters in the envelopes at random, one letter per envelope. What is the probability that exactly nine letters were inserted in the proper envelopes?

Q 2. Completely solve the following system of equations:

$$x + y + z + w = 10$$
$$x^2 + y^2 + z^2 + w^2 = 30$$
$$x^3 + y^3 + z^3 + w^3 = 100$$
$$xyzw = 24$$

Q 3. In a circle with center O and radius π the diameters AB and CD are perpendicular. On AB, $AE = \pi/7$. The perpendicular to AB at E meets the circle at M. From M a perpendicular is dropped to CD, meeting CD at F. How long is EF?

Q 4. Integrate $I = \int (32\cos^6 A - 48\cos^4 A + 17\cos^2 A - \sin^2 A)\,dA$.

Q 5. The happy Sunday vacationer, rented out-board motor wide open, had travelled one mile up a river when his hat blew off. Unconcerned he continued his trip, but ten minutes later he remembered that his return railroad ticket was under the hat band. Turning around immediately he recovered the hat opposite his starting point. How fast was the river flowing?

ANSWERS

A 1. If nine letters are in the correct envelopes, the tenth must be also, so the probability is zero.

A 2. By inspection (1,2,3,4) is a solution of the first and fourth equations, and satisfies the second and third equations. Since the equations are symmetrical in x,y,z,w, the other 23 permutations of 1,2,3,4 are solutions also. But these are all the solutions, since the product of the degrees of the equations is 4!

A 3. $EMFO$ is a rectangle so $EF = OM = \pi$.

A 4. The integrand reduces immediately to $\cos 6A$, so $I = (\sin 6A)/6$.

A 5. With reference to the river, the hat was stationary. The vacationer was without the hat for twenty minutes. During this time the river flowed one mile, at the rate of 3 miles per hour.

MATHEMATICAL MISCELLANY

Edited by

Charles K. Robbins, Purdue University

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: CHARLES K. ROBBINS, Department of Mathematics, Purdue University, Lafayette, Indiana.

Letter of the Month:

"Your new policy of having contributors write forewards to articles is very good. For those of us who are not professional mathematicians and not familiar with the more advanced phases it does help to clear up some of the mystery of what they are all about. I personally believe that mathematicians should clarify and simplify some of this material and thus make it more accessible to a larger number of people."

Very truly yours,

Ben F. Laposky

Cherokee, Iowa.
(Received by Professor Glenn James with permission to quote).

Instructor in algebra:	"In this complex fraction, why did you cancel these two expressions?"
Student:	"It's right, isn't it?"
Instructor:	"Yes."
Student:	"Don't tell me there's a reason!"

". . . Weeks on end the house was under the spell of a complicated figure consisting of not less than eight circles, large and small, and several engaged triangles, the whole to be drawn free-hand without lifting the pen — or, as a further refinement, to be drawn blindfold. Lawyer Paravant, the virtuoso of this kind of mental concentration, finally succeeced in performing the feat, perhaps with some loss of symmetry; but he was the only one.

We know on authority of the Hofrat that Lawyer Paravent studied mathematics; we know too the disciplinary grounds of his devotion to that branch of learning, and its virtue in cooling and dulling the edge of fleshly lusts . . . . There was one problem to which day and night he devoted all his brains, all the sporting pertinacity which once— before the beginning of this prolonged and enforced holiday, that even threatened at times to end in total quiscence — had gone to the convicting of criminals. It was — the squaring of the circle.

In the course of his studies, this retired official had convinced himself that the arguments on which science based the impossibility of the proposition were untenable; and that an over-ruling providence had removed him, Paravant, from the world of the living, and brought him here, having selected him to transfer the problem from the realms of the transcendental into the realms of the earthly and exact. By day and night he measured and calculated; covered enormous quantities of paper with figures, letters, computations, algebraic symbols; his face, which was the face of an apparently sound and vigorous man, wore the morose and visionary stare of a monomaniac; while his conversation, with consistent and fearful monotony, dealt with the proportional number pi, that abandoned fraction which the debased genius of a mathematician named Zachariah Dase one day figured out to the two-hundredth decimal place — purely for the joy of it and as a work of supererogation, for if he had figured it out to the two-thousandth, the result, as compared with unattainable mathematical exactitude, would have been practically unchanged. Everybody shunned the devoted Paravant like the plague; for whomever he succeeded in buttonholing, that unhappy wretch had to listen to a torrent of red-hot oratory, as the lawyer strove to arouse his humaner feelings to the shame that lay in the defilement of the mind of man by the hopeless irrationality of this mystic relation. The fruitfulness of forever multiplying the diameter of the circle by pi to find its circumference, of multiplying the square of the radius by pi to find its area, caused Lawyer Paravant to be visited by periodic doubt whether the problem had not been unnessarily complicated since Archimedes' day; whether the solution were not, in actual fact, a child's affair for simpleness. Why could not one rectify the circumference, why could not one also convert every straight line into a circle? Lawyer Paravant felt himself, at times, near a revelation. He was often seen, late in the evening, sitting at his table in the forsaken and dimly lighted dining-room, with a piece of string laid out before him, which he carefully arranged in circular shape, and then suddenly, with an abrupt gesture, stretched out straight; only to fall thereafter, leaning on his elbows, into bitter brooding. The Hofrat sometimes lent him a helping hand at the sport, and generally encouraged him in his freak. And the sufferer turned to Hans Castorp too, again and yet again, with his cherished grievance, finding in the young man much friendly understanding and a sympathetic interest in the mystery of the circle. He illustrated his pet despair to the young man by means of an exact drawing, executed with vast pains, showing a circle between two polygons, one inscribed, the other circumscribed, each polygon being of an infinite number of tiny sides up to the last human possibility of approximation to the circle. The remainder, the surrounding curvature, which in some ethereous, immaterial way refused to be rationalized by means of the calculable bounding lines, that, Lawyer Paravant said, with quivering jaw, was pi. Hans Castorp, for all his receptivity, showed himself less sensitive to pi than his interlocuter. He said it was all hocus-pocus; and advised Paravant not to over-heat himself with his cat's cradle; spoke of the series of dimensionless points of which the circle consisted, from its beginning — which did not exist — to its end — which did not exist either; and of the overpowering melancholy that lay in eternity, forever turning on itself without permanence of direction at any given moment — spoke with such tranquil resignation as to exert on Lawyer Paravant a momentary beneficent effect."

Thomas Mann, THE MAGIC MOUNTAIN, N.Y. 1927, ch. 7, Alfred A. Knopf, Inc. Publishers.

Dear Professor Trigg:

It took 43 days for your letter to reach my hand, and it took me one night to decide whether I should subscribe for your magazine or not. In the first place, I am just a young man of 32 who is interested in higher mathematics, but has very little background in it. In Thailand there are very few textbooks on higher mathematics. You can't find one in the National Library. My particular interests are geometry and theory of numbers. The only book I have on theory of numbers is Dickson's *Introduction to the Theory of Numbers.*

In the second place, it costs me all my pocket-money for the whole month, due to the high rate of exchange and the low salary paid by the government. But, I cannot disappoint you, so enclosed here please find a draft of 3 dollars, a subscription for one year of Mathematics Magazine.

As editor of the Problem Department, you may not wish to send me problems from other magazines, but as a mathematician, I think you will be glad to send some interesting problems in other magazines to me, especially the ones which are still unsolved.

I solved the three enclosed problems in about four hours, and think of the time to send the problems back and forth.

Yours sincerely,
Prasert Na Nagara

College of Agriculture,
Bangkhen, Bangkok, Thailand.

(Note to the Editor: I have forwarded a number of unsolved problems from other magazines to Mr. Nagara. Should any of our readers care to send any surplus advanced texts directly to him, it would be a gracious gesture which he will doubtless appreciate. Mr. Nagara has granted permission to publish his letter. — C. W. Trigg)

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

A Short Course in Differential Equations. By Earl D. Rainville, The Macmillan Co., New York, 1949, ix 210 pages, $3.00.

The scope of this text has been limited deliberately to a consideration of ordinary differential equations of the first order, linear differential equations with constant coefficients and special equations of order two. One chapter is devoted to definitions, another to hyperbolic functions. Three chapters cover first order, first degree equations. In the remaining seven chapters linear equations with constant coefficients, operational methods, applications to mechanical and electrical circuits, miscellaneous methods (variation of parameters and other methods) and special equations of order two are discussed.

This book has several interesting features. If, for instance, the solution of a differential equation is part of a circumference, not all of it, the restrictions are indicated. Mechanical and electrical systems are chosen as examples to which the methods of linear differential equations may be applied. References are restricted to a limited number of books and articles available in almost all college libraries. There is a large selection of problems. The presentation of the material on circuits appears to justify the assertion of the author: "...the student will benefit more from the detailed study of two important applications than he would from a casual contact with numerous separate ones."

William E. Byrne

THE CHRONOLOGY OF PI

Herman C. Schepler

(Continued from the January-February issue)

825. Muhammed Ibn Musa al-Khowarizmi (Alchwarizmi, Alkarism) (813-833) (Arabia) gave (used ?) 22/7 (3.1428 ⋯), $\sqrt{10}$ (3.162277 ⋯). and 62832/20000 (3.1416). The first he described as an approximate value, the second as used by geometricians, and the third as used by astronomers. He was an astronomer and the greatest mathematician at the court of al-Mâmûm; wrote Liber Algorism[6] (the book of al-Khowarizmi) the title of which gave the name to algebra. His name means Mohammed, the son of Musa (or Moses), from Khwarizm (Khiva), a locality south of the Black Sea. It was an Arab custom to give a man the name of his city. [1], 342; [5], 94, 102; [8], 298; [20], 28, 144; [22], 111; [26], 17.

850. Mahāvīra (India). Used $\sqrt{10}$. (3.162277 ⋯). His rule for the sphere is interesting, the approximate value being given as $\frac{9}{2}\left(\frac{1}{2}d\right)^3$ and the accurate value as $\frac{9}{10}\cdot\frac{9}{2}\left(\frac{1}{2}d\right)^3$, which means that π must be taken as 3.0375. He is perhaps the most noteworthy Hindu contributor to mathematics, possibly excepting Bhāskara, who lived three centuries later.

First half of 11th century. Frankos Von Luttich (Germany). No value available. He is claimed to have "contributed the only important work in the Christian era on squaring the circle." His works are published in six books, but only preserved in fragments. [22], 111.

1150. Bhāskara (Bhāskara Acharya) (1114 - c. 1185) (India) gave 3927/1250 (3.14160) as an "accurate" value, 22/7 (3.1428 ⋯) as an "inaccurate" value, and $\sqrt{10}$ (3.162277 ⋯) for ordinary work. The value 3927/1250 was possibly copied from Arya-Bhata, but is said to have been calculated afresh by Archimedes' method from the perimeter of a regular polygon of 384 sides. He also gave 754/240 (3.141666 ⋯), the origin of which is uncertain. Bhāskara wrote chiefly on astronomy and mathematics. He and Śrīndhara are the only outstanding writers in the history of Hindu mathematics from 1000 to 1500 A.D. [1], 341; [5], 85, 87; [8], 301; [20], 18.

1220. Leonardo Pisano (Leonardo of Pisa) (c. 1170 - c. 1250) (Italy), called Fibonacci, i.e., son of Bonaccio. 1440/(458 1/3) (3.1418 ⋯). His *Practica geometriae* refers to Euclid, Archimedes, Heron, and Ptolemy.

(6) The 14th century was characterized by the production of "algorisms", works devoted to the exposition of the uses of Hindu-Arabic numerals. The name is a corruption of al-Khowarizmi, but the books had no other connection with his work.

Without making use of Archimedes, he determined 1440/(458 1/5) (3.1427 ···) and 1440/(458 4/9) (3.1410 ···) from the regular polygon of 96 sides, of which he took the mean 1440/(458 1/3) (3.1418 ···). He was considered by some as the greatest mathematical genius of the Middle Ages. He traveled extensively and brought back to Italy a knowledge of the Hindu numerals and the general learning of the Arabs, which he set forth in many of his writings. [1], 342; [5], 120; [8], 218; [20], 24, 188; [24], 395.

1260. Johannes Campanus (Rome) 22/7 (3.1428571 ···). Sometime chaplain to Urban IV who reigned as pope from 1261 to 1264; published an edition of Euclid's *Elements*. [7], 31; [20], 25; [26], 25.

1430. Al Kashi (Arab) of Samerkand (Persia). 16 places; 3.1415926535898732. Wrote a short treatise in Persian·on arithmetic and geometry; was assistant to Ulugh Beg, the Royal Astronomer of Persia. [5], 108; [11], 261.

1460. Georg von Peurbach (Purbach) (1423-1461) (Austria). 62832/20000 (3.141600). Published in 1541. Studied under Nicholas de Cusa and other great teachers. Interested in astronomy and trigonomy and wrote an arithmetic. [1], 342; [5], 131; [8], 316; [20], 27.

1464. Nicolaus de Cusa (Nicolaus Cusanus) (1401-1464) (Germany). $(3/4) \times (\sqrt{3} + \sqrt{6}) = (3.1423 \cdots)$. Letters from Regiomontanus in 1464 and 1465, and published in 1533, rigidly demonstrated that de Cusa's quadrature was incorrect. This date was taken from these letters and may well have been earlier. (See 1464, Regiomontanus). de Cusa was a theologian, physicist, astronomer and geometer. The son of a fisherman, he rose rapidly in the Church, was made a cardinal, and became governor of Rome in 1448. His name was derived from his birthplace, the small town of Cues. [1], 342; [5], 143; [7], 47; [8], 315; [22], 111; [26], 28.

1464. Regiomontanus (Johann Müller) (1436-1476) (Germany). 3.14343. From his letters to de Cusa, published in 1533, proving de Cusa's results wrong. (See 1464, Nicolaus de Cusa). He was a pupil and associate of Peurbach; was a mathematician, astronomer, geographer, translator of Greek mathematics, and author of the first textbook of trigonometry. Pope Sixtus IV summoned him to Rome to aid in the reform of the calendar. He died there in 1476. Johannes Muller was commonly called Regiomontanus for his birthplace at Konigsberg, that is, king's mountain, or in Latin, regius mons. [1], 342; [8], 317; [20], 27, 296; Encyclopedia Brittanica.

1503. Tetragonismus. (Italy?). 22/7 (3.1428571 ···). *Circuli quadratura per Campanum, Archimedem Syracusanum* ··· . This book and the one mentioned in the next item, are probably the earliest in print on the subject of the quadrature. The quadrature of Campanus takes the ratio of Archimedes, (3.1428571···) to be absolutely correct. (See c. 1260, Campanus). [7], 31.

1503. Charles Bouelles (Paris, France) (3.1423 ···) announced anew the construction of de Cusa. (See c. 1464, Nicolaus de Cusa). The publication is: *In hoc opere contenta Epitome ··· Liber de quadratura Circuli* ···. Paris, 1503, folio. Was professor of theology at Noyon. Wrote on geometry and the theory of numbers, and was the first to give scientific consideration to the cycloid. [7], 31, 44; [18], 16; [22], 112.

1525. Stifel (1486-1567) (Germany). 3 1/8. *Underweysung,* etc. "The quadrature of the circle is obtained when the diagonal of the square conatins 10 parts of which the diameter of the circle contains 8." (Stated to be only approximate). Stifel's work had wide circulation in Germany. He was a minister who predicted the end of the world to occur October 3, 1533. Many of his followers who believed him spent and disposed of all their worldly goods and were ruined. He was imprisoned. [2], 220; [8], 221.

1544. Oronce Fine (Orontius Finaeus) (1494-1555) (Paris, France). His quadrature was disproved by Portugese Petrus Nonius (Pedro Nunes) (1502-1578) and also by Jean Buteo (1492-1572). He became professor of mathematics at College de France in 1532. [5], 142; [7], 50; [22], 112.

1573. Valentin Otto (Valentine Otho) (Germany). 355/113 (3.14159292035). He was an engineer. (See 480 A.D., Tsu Ch'ung chih, and 1585, Adriaen Anthonisz). [5], 73, 132; [23], 553.

1579. Francois Vieta (1540-1603) (Paris, France). Between 3.1415926535 and 3.1415926537. The first to give an infinite series:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

Following the Greek method, he considered polygons of $6 \cdot 2^{16}$ sides, i.e. 393,216 sides, and found π correct to nine places. He disproved the quadrature of Joseph Scaliger. Vieta was the greatest French mathematician of the 16th century. In 1580 he became master of requests at Paris, and later a member of the king's privy council. He wrote chiefly on algebra but was interested in the calendar and mathematics in general. [1], 342; [5], 143; [8], 321; [11], 261; [18], 16; [20], 38; [24], 395.

1580. Tycho Brahe (1546-1601) (Copenhagen, Denmark). $88/\sqrt{785}$ (3.1408 ···). Danish astronomer, his observatory being near Prague. Taught Johann Kepler. Brahe lost his nose in a duel with Passberg and adopted a golden one which he attached to his face with a cement that he always carried with him. [5], 159; [20], 188.

1585. Simon van der Eycke (Netherlands?) 3.1416055. In 1584 he gave 1521/484 equals 3.14256···. In 1585, van Ceulen gave π<3.14205<1521/484, found by calculating a regular polygon of 192 sides. In his reply, van der Eycke determined $\pi = 3.1416055$, whereupon van Ceulen in 1586 computed π between 3.142732 and 3.14103, and finally computed it correctly to 35 places. (See 1596, van Ceulen, and 1610, van Ceulen). [8], 222; [22], 112.

1585. Adriaen Anthonisz (Peter Metius) (1527-1607) (France) 355/113 (3.14159292 ⋯). Rediscovered this Chinese value, also known to the Japanese, when he was obliged to seek a still more accurate value than 3 1/7 to disprove the quadrature of van der Eycke. (See 1585, Simon van der Eycke). He proved that π lay between 377/120 and 333/106 from which he concluded that the true fractional value would be obtained by taking the mean of the numerators and the mean of the denominators of these fractions, which gave 355/113. The "rediscovery" was perhaps a lucky guess. The value was published in 1625 by his son Adriaen (1571-1635), who, from the fact that his family was originally from Metz, took the name of Metius. Adriaen Anthonisz also disproved the quadrature of Quercu. (See 480 A.D., Tsu Ch'ung-chih, and 1573, Velentin Otto). [1], 342; [5], 73, 80, 143; [8], 314; [18], 16; [22], 112; [24], 395.

1593. Adriaen van Rooman (Adrianus Romanus) (1561-1615) (Antwerp, Netherlands). He carried the computation to 17 places (15 correct), by computing the circumference of a regular circumscribed polygon of 2^{30} sides, i.e., 1,073,741,824 sides. He, Vieta and Clavius each disproved the quadrature of Joseph Scaliger. Romanus was successively professor of medicine and mathematics in Louvain, professor of mathematics at Würzburg, and royal mathematician (astrologer) in Poland; was the first to prove the usual formula for $\sin(A+B)$. [1], 342; [2], 230; [5], 143; [22], 113; [24], 395.

1596. Ludolph Van Ceulen (1540-1610) (Germany). To 20 places. This result was calculated from finding the perimeters of the inscribed and circumscribed regular polygons of $60 \cdot 2^{33}$ sides, i.e., 515,396,075,520 sides, obtained by the repeated use of a theorem of his discovery equivalent to the formula: $1 - \cos A = 2\sin^2 \frac{1}{2}A$. (See 1610, Ludolph Van Ceulen). [1], 342.

1610. Ludolph Van Ceulen (1540-1610) (Germany). To 35 places. π has since been called the Ludolphian Number in Germany. Van Ceulen devoted a considerable part of his life to the subject. His work was considered so extraordinary that the numbers were cut on his tombstone (now lost) in St. Peter's churchyard, at Leyden.

His post-humous arithmetic, published in Leyden, 1615, contains the result to 32 places, calculated from the perimeter of a polygon of 2^{62} sides, i.e., 4,611,686,018,427,387,904 sides. He also compiled a table of the perimeters of various regular polygons. His investigations led Snellius, Huygens, and others to further studies. (See 1585, Simon van der Eycke, and 1596, Ludolph Van Ceulen). [1], 342; [5], 143; [6], 10; [8], 222, 321; [18], 16; [20], 188; [24], 395.

1621. Snell (Willebrod Snellius) (1580-1626) (Leyden, Germany). To 34 places, published in his *Cyclometricus*, Leyden, 1621. He showed how narrower limits may be obtained for π without increasing the number of sides of the polygons by constructing two other lines from their sides which gave closer limits for the corresponding arcs. His method was so superior to that of Van Ceulen that the 34 places were obtained from a polygon of 2^{30} sides, i.e., 1,073,741,824 sides, from which Van Ceulen had obtained only 14 or perhaps 16 places. Similarly, the value for π obtained correct to two places by Archimedes from a polygon of 96 sides was obtained by Snell from a hexagon, while he determined the value correct to seven places from a polygon of 96 sides. Snell was a physicist, astronomer, and contributor to trigonometry. He discovered the law of refraction in optics in 1619. [1], 344; [2], 256; [5], 143; [7], 75; [8], 319.

1630. Grienberger. (Rome). To 39 places. Was among the last to make a calculation by the method of Archimedes. (See 1654, Christian Huygens). [1], 345; [6], 11.

1647. Gregory St. Vincent (1584-1667) (Belgium) a Jesuit, proposed four methods of squaring the circle in his book, *Opus geometricum quadraturae circuli et sectionum coni*, Antwerp, 1647, but they were not actually carried out. The fallacy in the quadrature was pointed out by Huygens and the work was attacked by many others. He published another book on the subject in 1668. Montucla remarks that "no one ever squared the circle with so much ability or (except for his principal object) with so much success". His *Theoremata Mathematica*, published in 1624, contains a clear account of the method of exhaustions, which is applied to several quadratures, notably that of the hyperbola. He discovered the property of the area of the hyperbola which led to Napier's logarithms being called hyperbolic. [2], 309; [5], 181; [8], 318.

1647. William Oughtred (1574-1660) (England), designated the ratio of circumference of a circle to its diameter by π/δ. π and δ were not separately defined, but undoubtedly π stood for periphery and δ for diameter. Oughtred's notation was adopted by Isaac Barrow (publication date, 1860) and by David Gregory (1697) except that he writes π/ρ, ρ being the radius. Oughtred wrote on arithmetic and trigonometry. [4], 8.

1650. John Wallis (1616-1703). (England). Unlimited series. By an extremely difficult and complicated method he arrived at the interesting expression for π:

$$\frac{4}{\pi} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots} \quad (7)$$

(7) It is often given in another form:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}$$

He showed this value to Lord Brouncker (1620-1684), first president of the Royal Society, who brought it into the form of a continued fraction:

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}}$$

Wallis was Savilian professor of geometry at Oxford and published many mathematical works. In his *Arithmetica infinitorum*, the square ◻ stands for 4/3.14149 ··· . In 1685 he represented by π the "periphery" described by the center of gravity in a revolution. [1], 345; [4], 8; [5], 186; [6], 11; [8], 321; [11], 261; [12], 76; [20], 188; [26], 74.

1654. Christian Huygens (1629-1695). (Hague, Netherlands). 9 places. Used only the inscribed polygon of 60 sides. Last noteworthy attempt made by Greek methods. He proved Snell's theorems and made the greatest refinements in the use of the geometrical method of Archimedes. (See 1621, Snell). With his labors the ancient methods may be said to close. Famous physicist and astronomer, and made numerous contributions to mathematics, particularly to the study of curves. [5], 143; [8], 310; [23], 424; [24], 395.

1666. Thomas Hobbes. (Malmsbury and London, England.) 3 1/5 (3.2) Refuted by Huygens and Wallis. In 1678 he published ···*Proportion of a Straight Line to Half the Arc of a Quadrant* in London, in which $\sqrt{10}$ (3.16227 ···) was given. Celebrated English philosopher. [7], 109; [9], 15; [21], 136; [22], 114.

1666. Satō Seikō (Japan). 3.14. Given in his *Kongenki*. First Japanese work in which the ancient Chinese method of solving numerical higher equations appears. [5], 79.

1668. James Gregory (1638-1675) (Scotland). Unlimited series: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. When $x = 1$, the series becomes $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. This series was discovered independently by Leibnitz in 1673. See 1673, Leibnitz.(8)

Gregory proved the geometrical quadrature of the circle impossible in 1668. In 1661 he invented, but did not practically construct the telescope bearing his name. He originated the photometric method of estimating the stars. After living in Italy for some years he returned

(8)The series is also given as:

$$\frac{\pi}{4} = 1 - 2\left(\frac{1}{3\times 5} + \frac{1}{7\times 9} + \frac{1}{11\times 13} + \frac{1}{15\times 17}\cdots\right)$$

to Scotland in 1668 where he became professor of mathematics at St. Andrews; and at Edinburgh in 1674. [5], 143; [6], 11; [7], 118; [8], 308; [11], 261; [24], 396.

1673. Gottfried Wilhelm von Leibniz (Leibnitz) (1646-1716) (Leipzig, Germany) Series: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \cdots$. This series had previously been discovered by Gregory. (See 1668, Gregory). Leibniz was the only first-class pure mathematician produced by Germany during the 17th century. He shares credit with Newton in developing the differential and integral calculus. [5], 206; [8], 312; [20], 188.

1680. Isaac Newton (1642-1727) (England) significantly gave no value for π. Began his work on the calculus in 1665 and was considered the most promising mathematician in England in 1668. In 1669 he succeeded Barrow as Lucasian professor of mathematics at Cambridge. Contributed extensively to all branches of mathematics then known, particularly to the theories of series, equations, and curves. [5], 202; [8], 315.

1685. Father Kochansky (Poland) 3.1415333 ··· by geometrical construction. *An Approximate Geometrical Construction for Pi*, Leipsiger Berichte, 1685. This construction passed into many geometrical textbooks. Kochansky was librarian of the Polish King John III. In Figure 1, let *AB* be the diameter of the circle. Construct perpendiculars to this diameter at extremities *A* and *B*. Construct the 30° angle *DOB*; *D* is the intersection of *OD* with the perpendicular at *B*. Mark off *AC* equal to three times the radius of the circle. Line *CD* is the approximate semi-circumference. The error is -.0000593. [6], 14; [18], 22; [21], 237; [25], 40.

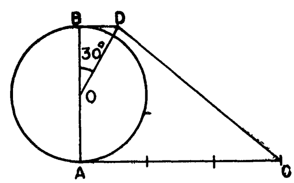


Fig. 1

1690. Takebe (Takebe Hikojirō Kenkō) (1664-1739) (Japan). Unlimited series. Correct to 41 figures. [11], 261; [23], 553.

1699. Abraham Sharp (1651-1742) (England) calculated π to 72 decimal places (71 correct) under instructions from E. Halley. The value was obtained by taking $x = \sqrt{\frac{1}{3}}$ in Gregory's series, giving

$$\frac{\pi}{6} = \sqrt{\frac{1}{3}}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \cdots\right)$$

which is more usable than the form of the series giving $\frac{\pi}{4}$ when $x = 1$. (See 1668, James Gregory). [1], 346; [5], 206; [12], 77; [18], 16; [24], 397.

18th century. Oliver de Serres. (France). 3. With a pair of scales he determined that a circle weighed as much as the square upon the side of the equilateral triangle inscribed in it, which gives $\pi = 3$. [22], 115.

1706. John Machin (1680-1752) (England). 100 places. Obtained by substituting Gregory's infinite series for arc tan (1/5) and arc tan (1/239) in the expression: $\frac{\pi}{4} = 4$ arc tan (1/5) - arc tan (1/239). He was a mathematician and professor of astronomy at Gresham College, London. [5], 206; [18], 16; [24], 397.

1706. William Jones (1675-1749) (England) designated the ratio of the circumference of a circle to its diameter by π. This is the first occurrence of the sign π for the ratio. [4], 9.

1713. Anonymous. (China). 19 figures. In the *Su-li Ching-yün,* compiled by Imperial order in 1713. [24], 394.

1719. Thomas Fantet de Lagny (Lagny) (1660-1734) (France) 127 places (112 correct). He was a French mathematician. [1], 346; [5], 203; [24], 397.

1720. Matsunaga (Matsunaga Ryōhitsu) (Japan). Correct to 50 figures in our notation. [11], 261; [23], 553.

1728. Malthulon (France) offered solutions to squaring the circle and to perpetual motion. He offered 1000 crowns reward in legal form to anyone proving him wrong. Nicoli, who proved him wrong, collected the reward and abandoned it to the Hotel Dieu of Lyons. Later the courts gave the money to the poor. [6], 14; [17], 18; [22], 114.

1728. Alexander Pope (1688-1744) (England) Poet. In his *Dunciad* is mentioned, "The mad Mathesis, now, running round the circle, finds it square." A note explains that this "regards the wild and fruitless attempts of squaring a circle." [18], 12; Americana Encyclopedia.

1753. M. de Causans, of the Guards (France) 4. He cut a circular piece of turf, squared it, and deduced original sin and the Trinity from the result; found that the circle was equal to the square in which it is inscribed, making $\pi = 4$. He offered a reward for the detection of any error and actually deposited 10,000 francs as earnest of 300,000, but the courts would not allow anyone to recover. [18], 11.

1754. Montucla (1725-1799) (France). *Histoire des rechurches sur la quadrature du cercle ⋯*. Paris. This was *the* history of the subject in its time and is still a classic on its early history. [7], 159; [23], 540.

1755. The French Academy of Sciences (France) refused to examine any more quadratures or problems of similar nature. De Morgan called this "the official blow to circle-squarers." The Royal Society, London, followed suit a few years later. [5], 246; [7], 163.

1760. Count (Comte) de Buffon (1707-1788) derived his famous *Needle Problem* in which π is determined from probability.(9) A number of equidistant parallel straight lines, distance a apart, are ruled upon a plane surface. A stick of length L, which is less than a, is dropped on the plane. The probability of it falling so as to lie across one of the lines is $2L/\pi a$. π may be evaluated by repeating the experiment a large number of times. When $L = a$, the probability has the simplified value of $2/\pi$. Of the many experiments that have been performed by this method, perhaps the most accurate determination of π was made by Lazzerini, an Italian mathematician, in 1901. He made 3,408 tosses of a needle across parallel lines, giving a value for π of 3.1415929, an error of only .000,000,3. Other similar methods of approximating the value of π have been employed at various times. [1], 348; [5], 243; [6], 17; [7], 170; [12], 246; [22], 118.

1761. Johann Heinrich Lambert (1728-1777) (Germany) proved that π is irrational, i.e., not expressible as an integer or as the quotient of two integers. A physicist, mathematician and astronomer; founder of hyperbolic trigonometry. [5], 2, 246; [8], 312.

1776. Hesse (Berlin, Germany). 3 14/99 (3.14141). Wrote an arithmetic in which this value was given. [21], 136.

1776. Hutton. (1737-1823) (England). Suggested the use of the formula:

$$1/4\ \pi = \tan^{-1} 1/2 + \tan^{-1} 1/3, \quad \text{or}$$

$$1/4\ \pi = 5\ \tan^{-1} 1/7 + 2\ \tan^{-1} 3/79.$$

This formula was also suggested by Euler in 1779, but neither Hutton nor Euler carried the approximation as far as had been done previously by other formulae. Hutton was one of the best known English writers on mathematics at the close of the 18th century; was professor of mathematics at the Military Academy at Woolrich (1772-1807).

1779. Leonard Euler (Leonhard Euler) (1707-1783) (Basel, Switzerland). Series. Published in 1798. $\pi = 20 \text{ arc tan } (1/7) + 8 \text{ arc tan } (3/79)$. Euler used p for the circumference-to-diameter ratio in 1734, and c in 1736. He first used π in 1742. After the publication of his *Introductio in analysin infinitorum* in 1748, the use of the symbol π for the ratio became favorable for wider adoption. Euler used π in most of his later

(9) *Essai d'Arithmetique Moral* by Buffon, appearing in Vol. IV of the *Supplement a l'Histoire Naturelle,* in 1777.

publications. He taught mathematics and physics in Petrograd and was one of the greatest physicists, astronomers and mathematicians of the 18th century. [1], 346; [4], 11; [5], 227, 232; [8], 306; [20], 189; [24], 398.

1788. M. de Vausenville (France). Laboring under the erroneous impression that a reward had been offered for the solution of the quadrature, he brought action against the French Academy of Sciences to recover a reward to which he felt himself entitled. [18], 9.

1789. Georg Vega (1756-1802) (Austria). 143 places (126 correct). [1], 346.

1794. Georg Vega (1756-1802) (Austria). 140 places (136 correct) (See 1789, Georg Vega). [1], 346.

1794. Adrien Marie Legendré (1752-1833) (France) used π in his *Éléments de géométrie* as a symbol for the circumference/diameter ratio. This was the earliest elementary French schoolbook to contain π in regular use. The proofs of the irrationality of π and π^2 are also given in this publication. Legendré was a celebrated mathematician who contributed to the theory of elliptic functions, the theory of numbers, least squares, and geometry. [4], 13.

1797. Lorenzo Mascheroni (1750-1800) (Italy) proved that all constructions possible with ruler and compasses are possible with compasses alone; claimed that constructions with compasses are more accurate than those with ruler. [5], 268; [8], 313.

End of 18th Century. Anonymous. F. X. von Zach (in England) saw a manuscript by an unknown author in the Radcliffe Library, Oxford, which gives the value of π to 154 places (152 correct). [1], 346.

1825. Malacarne. (Italy). Less than 3. One of the last to attempt the geometrical quadrature. His *Solution Géométrique* was published at Paris, 1825. [7], 118.

1828. Specht (Germany) 3.141591953 ··· by geometrical construction. Crelle's Journal,[10] Vol. III, page 83. "The rectangle with sides equal to AE and half the radius r is very approximately equal in area to the circle". In Figure 2, on the tangent to the circle at A, let $AB = 2\ 1/5$ radius and $BC = 2/5$ radius. On the diameter through A take $AD = OB$, and draw DE parallel to OC. Then $AE/AD = AC/AO = 13/5$. Therefore, $AE = r \cdot \frac{13}{5}\sqrt{1 + \left(\frac{11}{5}\right)^2} = r \cdot \frac{13}{25}\sqrt{146}$. Thus, AE =

[10] Complete title of the publication is *Fur die Reine und Angewandte Mathematik*, (Berlin).

$r \cdot 6.283183906 \cdots$, which is smaller than the circumference of the circle by less than two millionths of the radius. The error is $-.000000700 \cdots$. [10], 34.

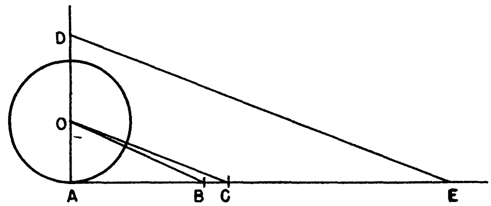


Fig. 2

1833. William Baddeley (England) 3.202216 24/361. *Mechanical Quadrature of the Circle,* London Mechanics' Magazine, August, 1833. "From a piece of carefully rolled sheet brass was cut out a circle 1 9/10 inches in diameter, and a square 1 7/10 inches in diameter. On weighing them they were found to be of exactly the same weight, which proves that, as each are of the same thickness, the surfaces must also be precisely similar. The rule, therefore, is that the square is to the circle as 17 to 19." This would make his ratio, 3.202216 24/361. [9], 5.

1836. LaComme (Paris, France). 3 1/8 (3.125). A French well sinker, requesting information from a mathematics professor regarding the amount of stone required to pave the circular bottom of a well, was told that a correct answer was impossible since the exact ratio of the diameter of a circle to its circumference had never been determined. This true but impractical statement set LaComme to thinking. After a haphazard study of mathematics he announced his discovery that 3 1/8 was the exact ratio. Although π had been accurately determined to 152 decimal places in the 18th century (see End of 18th Century, F. X. von Zach), LaComme was honored for his profound discovery with several medals of the first class, bestowed by Parisian societies. [7], 46; [18], 27.

1837. J. F. Callet (Paris, France). 154 places (152 correct). The result of this calculation of π was published in J. F. Callet's Tables in 1837. The calculation may have been made by someone else. [1], 346.

1841. William Rutherford. (England). 208 places (152 correct). He used the formula: $\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$. [1], 346; [18], 16.

1844. Zacharias Dase (1824-1861) (Hamburg, Germany). To 205 places (200 correct) using the formula: $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$. Dase was a lightning calculator employed by Gauss. [1], 346; [12], 77; [20], 188; [24], 398.

1847. Thomas Clausen (1801-1885) (Germany). 250 places (248 correct). His calculation was made independently by the formula:

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

[1], 346.

1849. Jakob de Gelder (Germany). 3.14159292035 ··· by geometrical construction. Grunert's Archiv., Vol. VII, 1849, p. 98. Since $\frac{355}{113} = 3 + \frac{4^2}{7^2+8^2}$, it can easily be constructed. In Figure 3, let $CD = 1$; $CE = 7/8$; $AF = 1/2$; and let FG be parallel to CD and FH parallel to EG. Then $AH = \frac{4^2}{7^2+8^2} = 0.14159292035\cdots$. The error is +.000000266···
[5], 73; [10], 34.

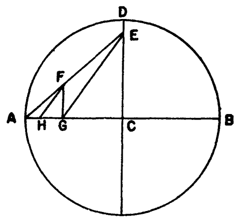


Fig. 3

1851. John Parker. 20612/6561 (3.141594269 ···). Published in his book *Quadrature of the Circle.*

1853. William Rutherford. (England). 440 places (all correct). (See 1841, Rutherford). [1], 346.

1853. William Shanks (1812-1882). (England). 607 places, using Machin's formula. (See 1706, John Machin). Shanks, English mathematician, was assisted by Dr. William Rutherford in the verification of the first 440 decimals of π. (See 1853, William Rutherford). [1], 346; [6], 12; [9], 20; [18], 16.

1853. Richter. (Germany?). 333 places (330 correct). This was presumably done in ignorance of what had been done in England. [1], 346.

1854. Richter. (Died, 1854) (Germany?). 500 places. This value was published in 1855. [1], 346; [12], 77; [20], 188; [23], 311; [24], 398.

1855. Athenaeum (publication, Oxford University Press). 4. A correspondent makes the square equal to a circle by making each side equal to a quarter of the circumference. [18], 12.

1860. James Smith (Liverpool, England). 3 1/8 (3.125). Published several books and pamphlets arguing for the accuracy of his value, and even attempted to bring it before the British Association for the Advancement of Science. Professors De Morgan and Whewell, and even the famous mathematician, Sir William Rowan Hamilton, tried unsuccessfully to convince him of his error. In a letter to Smith, Professor Whewell gave the following demonstration: "You may do this. Calculate the side of a polygon of 24 sides inscribed in a circle. You will find that if the radius of the circle be one, the side of the polygon is .264. Now the arc which this side subtends is, according to your proposition, 3.125/12 = .2604, and therefore, the chord is greater than its arc, which, you will allow, is impossible". [7], 46; [18], 28.

1862. Lawrence Sluter Benson (U.S.A.). 3.141592 ⋯. Endeavored to demonstrate that the area of the circle is equal to $3R^2$, or the arithmetical square between the inscribed and circumscribed squares. His theorem is: "The $\sqrt{12}$ = 3.4641016 + is the ratio between the diameter of a circle and the perimeter of its equivalent square". It was his belief that the ratio between the diameter and circumference is not a function of the area of the circle, but that the area of the circle is $\doteq 3R^2$, or .75. He accepted the value of π = 3.141592 + . He published some 20 pamphlets on the area of the circle, three volumes on philosophic essays, and one geometry. [9], 6.

1863. S. M. Drach (England). 3.14159265000 ⋯ Suggested an approximation for finding the circumference of a circle. Phil. Mag., Jan., 1863. The solution is extremely accurate but awkward to construct. From three diameters, deduct 8/1000 and 7/1,000,000 of a diameter, and add 5 per cent to the result. This gives a length which is smaller than the circumference by about 1 1/60 inch in 14,000 miles. The error is −.00000000358 ⋯. [9], 10; [18], 24.

1868. Cyrus Pitt Grosvenor, Rev. (New York). 3.142135 ⋯ by geometrical construction. Published in a pamphlet, *The Circle Squared,* New York, 1868. The following rule is given for the area: Square the diameter of the circle; multiply the square by two; extract the square root of the product; from the root subtract the diameter of the circle; square the remainder; multiply this square by five-fourths; subtract the product from the square of the diameter of the circle. In algebraic terms the rule is:

$$\text{Area} = D^2 - \frac{5}{4}(\sqrt{2D^2} - D)^2 = D^2 - \frac{5}{4}D^2(\sqrt{2} - 1)^2$$
$$= D^2[1 - \frac{5}{4}(\sqrt{2} - 1)^2] = D^2(0.7855339706472\cdots) = \frac{\pi D}{4}.$$

The error for the value of π from this construction is +.000543. [9], 14.

This article will be concluded in the May-June issue.

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Mr. J. Baker, P. O. Box 5962, Johannesburg, South Africa, desires very much to be contacted by someone who has and will sell the March '39 and April '44 issues of the National Mathematics Magazine.